Noise Kernel and Stress Energy Bi-Tensor of Quantum Fields in Conformally-Optical Metrics: Schwarzschild Black Holes

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In Paper II [N. G. Phillips and B. L. Hu, this journal] we presented the details for the regularization of the noise kernel of a quantum scalar field in optical spacetimes by the modified point separation scheme, and a Gaussian approximation for the Green function. We worked out the regularized noise kernel for two examples: hot flat space and optical Schwarzschild metric. In this paper we consider noise kernels for a scalar field in the Schwarzschild black hole. Much of the work in the point separation approach is to determine how the divergent piece conformally transforms. For the Schwarzschild metric we find that the fluctuations of the stress tensor of the Hawking flux in the far field region checks with the analytic results given by Campos and Hu earlier [A. Campos and B. L. Hu, Phys. Rev. D **58** (1998) 125021; Int. J. Theor. Phys. **38** (1999) 1253]. We also verify Page's result [D. N. Page, Phys. Rev. **D25**, 1499 (1982)] for the stress tensor, which, though used often, still lacks a rigorous proof, as in his original work the direct use of the conformal transformation was circumvented. However, as in the optical case, we show that the Gaussian approximation applied to the Green function produces significant error in the noise kernel on the Schwarzschild horizon. As before we identify the failure as occurring at the fourth covariant derivative order.

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I. INTRODUCTION

Many physically interesting metrics are conformally related to an optical metric. When the metric has an optical form, the Gaussian approximation [4] for the Green function [3] yields a closed analytic form. For a given noise kernel in an optical metric, one can take advantage of the simple conformal transformation property of the scalar field's Green function to compute the noise kernel for the corresponding conformally-optical metrics. In Paper II we have derived the regularized noise kernel in the optical metric by the modified point separation method and worked out two examples: hot flat space and optical Schwarzschild metrics. Hot flat space is related to thermal fields in the (spatially-flat) Robertson-Walker universe by a time dependent scale factor, and the optical Schwarzschild metric is of course conformally related to the Schwarzschild black hole metric. In this paper we present the details for computing the regularized noise kernel.

According to the procedure outlined in Paper II, the main obstacle here is the subtraction of the Hadamard ansatz. The divergent Green function is defined in terms of the optical metric while the Hadamard ansatz in terms of the physical metric. We need to re-express the transformed optical metric in terms of the physical metric. The defining equations for the geometric objects on the optical metric are transformed to the physical metric and solved recursively.

Now the Green function series expansion can be written solely in terms of the physical metric. While for the ultrastatic spacetimes considered in Paper II the Hadamard subtraction was straightforward, for spacetimes only conformally ultrastatic this subtraction is non-trivial. Before the subtraction can be carried out and the Green function regularized, we need to study the conformal transformation properties of the point separation objects used to define the Green function. As we will show, regularizing the divergent structure to sufficient order for the noise kernel is no small task, when we include the conformal transformation between the physical and the optical metrics. (e.g., the fourth order term in the expansion of the renormalized Green function has over 1100 terms). It is at this point that the symbolic computation environment takes over.

Since all the series expansions used are recursively derived, as a check of the full expression it is sufficient that we get the correct results for the lower order terms. Indeed our general expression contains and confirms the Page

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[3] result for the vacuum expectation value of the stress tensor in the Schwarzschild black hole. With the general expansion of the renormalized Green function at hand, one can choose a metric and compute the coincident limit of the noise kernel as outlined in Paper II. In Section II we compute the noise kernel for spacetimes conformally related to ultrastatic spacetimes. In Section III we specialize to the Schwarzschild black hole and discuss the significance of our results in Sec. IV. The Appendices contain formulas whose usefulness goes beyond the specific approximations adopted here.

II. NOISE KERNEL IN CONFORMALLY-OPTICAL SPACETIMES

For a static physical metric g_{ab}, we consider the conformally related optical metric

$$\bar{g}_{ab} = e^{-2\omega} g_a b \tag{2.1}$$

where the conformal factor $e^{-2\omega}$ is the space-dependent function such that $\bar{g}_{\tau\tau} = 1$, *i.e.*, the metric $\bar{g}_{ab}$ is for an ultrastatic spacetime. In general, we use the overbar to denote objects defined in terms of the optical metric $\bar{g}_{ab}$ and without for those in terms of the physical metric g_{ab}. For a conformally invariant field, the Green functions on the two spacetimes are related via

$$G(x,y) = e^{-\omega(x)} \bar{G}(x,y) e^{-\omega(y)}. \tag{2.2}$$

This is our starting point. In terms of the Gaussian approximation

$$\bar{G}_{\rm Gauss}(x,y) = \frac{\kappa \bar{\Delta^{1}}^{/2}}{8\pi^2 \bar{r}} \frac{\sinh \kappa \bar{r}}{(\cosh \kappa \bar{r} - \cos \kappa \tau)}, \tag{2.3}$$

where $r = \left(2\,{}^{(3)}\bar{\sigma}\right)^{\frac{1}{2}}$ and κ is the period of the imaginary time dimension, we have the expansion of the physical Green function as

$$\begin{aligned} G_{\rm Gauss} \;=\; & \frac{\bar{\Delta}^{\frac{1}{2}}\, e^{-\omega-\omega'}}{8\,\bar{\sigma}\,\pi^2} + \frac{\bar{\Delta}^{\frac{1}{2}}\, e^{-\omega-\omega'}}{8\,\pi^2} \left\{ \frac{\kappa^2}{6} + \frac{\kappa^4}{180} \left(-\bar{\sigma} + 2\,\Delta\tau^2\right) \right. \\ & \left. + \frac{\kappa^6}{3780} \left(\bar{\sigma}^2 - 6\,\bar{\sigma}\,\Delta\tau^2 + 4\,\Delta\tau^4\right) \right\} + O\left(\sigma^{\frac{5}{2}}, \Delta\tau^5\right) \end{aligned} \tag{2.4}$$

where $\omega = \omega(x)$ and $\omega' = \omega(y)$.

We proceed as in Paper II and subtract from $G_{\rm Gauss}$ the Hadamard ansatz

$$S(x,y) = \frac{1}{16\,\pi^2} \left(\frac{2\,\Delta^{\frac{1}{2}}}{\sigma} + \sigma\, w_1 + \sigma^2\, w_2 \right) + O\left(\sigma^3\right) \tag{2.5}$$

to get the renormalized Green function

$$G_{\rm ren} = G_{\rm gauss} - S. \tag{2.6}$$

In the Hadamard ansatz, the $V(x,x')$ term is not included since there is no $\log\sigma$ divergences present in the expansion of the Gaussian approximation to the Green function. This can also be viewed as an extension of the Gaussian approximation.

Now the situation here is different from the optical case as the divergent terms present (2.4) and (2.6) do not directly cancel. Much of the work for regularizing the Green function will entail showing that indeed the difference between these two divergent terms is finite and to develop this finite difference to sufficient order to compute the noise kernel. To this end, we write the renormalized Green function as

$$G_{\rm ren} = G_{\rm div,ren} + G_{\rm fin} - \frac{1}{(4\pi)^2} W \tag{2.7}$$

with

$$G_{\rm div,ren} \;=\; \frac{1}{8\pi^2} \left(\frac{\bar{\Delta}^{\frac{1}{2}}\, e^{-\omega-\omega'}}{\bar{\sigma}} - \frac{\Delta^{\frac{1}{2}}}{\sigma} \right) \tag{2.8a}$$

$$G_{\rm fin} \;=\; \frac{\bar{\Delta}^{\frac{1}{2}}\, e^{-\omega-\omega'}}{8\,\pi^2} \left\{ \frac{\kappa^2}{6} + \frac{\kappa^4}{180} \left(-\bar{\sigma} + 2\,\Delta\tau^2\right) \right.$$

$$+\frac{\kappa^6}{3780}\left(\bar\sigma^2 - 6\,\bar\sigma\,\Delta\tau^2 + 4\,\Delta\tau^4\right)\Bigg\} \tag{2.8b}$$

$$W \;=\; \sigma\, w_1 + \sigma^2\, w_2 \tag{2.8c}$$

Both $G_{\rm fin}$ and W have well behaved coincident limits. As these functions stand, it is only the last one, W, that we can readily handle. Appendix E of Paper II gives the series expansion in terms of the world function σ defined with respect to the physical metric g_{ab}. Since we want the noise kernel in the physical metric, it is the covariant derivative commensurate with this metric we must use when computing the noise kernel. On the other hand, the function $G_{\rm fin}$ is defined in terms of the world function and the VanVleck-Morette determinant of the optical metric $\bar g_{ab}$. Thus the coincident limit expressions for σ and the series expansion for $\Delta^{\frac12}$ derived in the Appendices of Paper II cannot be used to determine the contribution to a series expansion of the renormalized Green function $G_{\rm ren}$.

In Appendix A here we show how to get around this problem. Both $\bar\sigma$ and $\bar\Delta^{\frac12}$ are defined in terms of covariant differential equations with respect to the optical metric. The conformal transformation properties of the covariant derivative are used to re-express these equations in terms of the covariant derivative commensurate with the physical metric. Then end point series solutions built from the physical metric world function σ are found. The details, along with the found solution are collected in the Appendices of this paper. Using these results, we can determine the contribution from $G_{\rm fin}$ to the noise kernel.

This leaves the first function defined above to deal with. The key to unlocking this term is to introduce the symmetric function

$$\Sigma(x,y) = e^{\omega(x)}\bar\sigma(x,y)e^{\omega(y)} - \sigma(x,y) \tag{2.9}$$

The important property of this function as shown in Appendix A is $\Sigma \sim \sigma^2$ as $\sigma \to 0$. With this function, we have

$$G_{\rm div,ren} = \frac{1}{8\,\pi^2}\frac{\sigma\,\left(\bar\Delta^{\frac12} - \Delta^{\frac12}\right) - \Sigma\,\Delta^{\frac12}}{\sigma\,\left(\sigma + \Sigma\right)} \tag{2.10}$$

To see that this is indeed finite, we use the expansions of

$$\Sigma \;\sim\; \sigma^2\, S^{(4)} \tag{2.11a}$$

$$\Delta^{\frac12} \;\sim\; 1 + \sigma\,\Delta^{(2)} \tag{2.11b}$$

$$\bar\Delta^{\frac12} \;\sim\; 1 + \sigma\,\bar\Delta^{(2)} \tag{2.11c}$$

to obtain the leading order behavior

$$G_{\rm div,ren} = -\frac{1}{8\,\pi^2}\frac{-\bar\Delta^{(2)} + \Delta^{(2)} + S^{(4)} + \sigma\,\Delta^{(2)}\,S^{(4)}}{1 + \sigma\,S^{(4)}} \tag{2.12}$$

which is finite as $\sigma \to 0$ and has the value

$$[G_{\rm div,ren}] = \frac{1}{8\,\pi^2}\left(\bar\Delta^{(2)} - \Delta^{(2)} - S^{(4)}\right) \tag{2.13}$$

Using Eqns (A11a) and (A16a), along with (C8) of Paper II, we get the explicit form of the leading order expansion tensors used above as

$$\Sigma^{(4)}_{abcd} \;=\; \frac{\omega_{;c}\,\omega_{;d}\,g_{ab}}{12} + \frac{\omega_{;cd}\,g_{ab}}{12} - \frac{\omega_{;p}\,\omega^{;p}\,g_{ab}\,g_{cd}}{24} \tag{2.14a}$$

$$\Delta^{(2)}_{ab} \;=\; \frac{R_{ab}}{12} \tag{2.14b}$$

$$\bar\Delta^{(2)}_{ab} \;=\; \frac{\omega_{;a}\,\omega_{;b}}{6} + \frac{\omega_{;ab}}{6} - \frac{\omega_{;p}\,\omega^{;p}\,g_{ab}}{6} + \frac{\omega_{;p}{}^{p}\,g_{ab}}{12} + \frac{R_{ab}}{12} \tag{2.14c}$$

From these we can get the expansion scalars we need for (2.13):

$$S^{(4)} \;=\; 4p^p p^q p^r p^s \Sigma^{(4)}_{pqrs} = -\frac{(\omega_{;p}\,\omega^{;p})}{6} + \frac{\omega_{;p}\,\omega_{;q}\,p^p\,p^q}{3} + \frac{\omega_{;pq}\,p^p\,p^q}{3} \tag{2.15a}$$

$$\Delta^{(2)} \;=\; 2p^p p^q \Delta^{(2)}_{pq} = \frac{p^p\,p^q\,R_{pq}}{6} \tag{2.15b}$$

$$\bar\Delta^{(2)} \;=\; 2p^p p^q \bar\Delta^{(2)}_{pq} = -\frac{(\omega_{;p}\,\omega^{;p})}{3} + \frac{\omega_{;p}{}^{p}}{6} + \frac{\omega_{;p}\,\omega_{;q}\,p^p\,p^q}{3} + \frac{\omega_{;pq}\,p^p\,p^q}{3} + \frac{p^p\,p^q\,R_{pq}}{6} \tag{2.15c}$$

where p^a is a unit vector. From these expressions, we might expect there to be residual direction dependence for (2.13). But when we substitute the expansion scalars into the $\sigma \to 0$ value of $G_{\rm div,ren}$, the direction dependences of the three expansion scalars cancel and we are left with

$$[G_{\rm div,ren}] = \frac{1}{48\,\pi^2}\left(\omega_{;p}{}^{p} - \omega_{;p}\,\omega^{;p}\right) \tag{2.16}$$

Using $\left[\bar{\Delta}^{\frac{1}{2}}\right] = 1$ and $[\bar{\sigma}] = 0$, we can immediately get the coincident limit of (2.8b)

$$[G_{\rm fin}] = \frac{\kappa^2}{48\,e^{2\,\omega}\,\pi^2} \tag{2.17}$$

Since $[W] = 0$, we readily obtain the coincident limit of the renormalized Green function

$$[G_{\rm ren}] = \frac{1}{48\,\pi^2}\left(\frac{\kappa^2}{e^{2\,\omega}} - \omega_{;p}\,\omega^{;p} + \omega_{;p}{}^{p}\right), \tag{2.18}$$

which is the result derived by Page (Eq. (29) of [3]).

Now that we know we can regularize $G_{\rm div,ren}$, we turn to developing the series expansion of $G_{\rm div,ren}$ to a sufficient order to compute the coincident limit of the noise kernel. We need the expansion

$$G_{\rm div,ren} = \frac{1}{8\,\pi^2}\left(G^{(0)}_{\rm div,ren} + \sqrt{\sigma}\,G^{(1)}_{\rm div,ren} + \sigma\,G^{(2)}_{\rm div,ren} + \sigma^{\frac{3}{2}}\,G^{(3)}_{\rm div,ren} + \sigma^2\,G^{(4)}_{\rm div,ren}\right) \tag{2.19}$$

The computation of $[G_{\rm div,ren}]$, i.e. $G^{(0)}_{\rm div,ren}$, involves Σ to order σ^2 and both $\Delta^{\frac{1}{2}}$ and $\bar{\Delta}^{\frac{1}{2}}$ to order σ. Thus to carry out the expansion (2.19) we will need these functions expanded to order σ^4 for Σ and σ^3 for $\Delta^{\frac{1}{2}}$ and $\bar{\Delta}^{\frac{1}{2}}$. These are done in Appendix A. For this section we will use expansions in terms of the scalar coefficients:

$$\begin{aligned}
\Sigma &\sim \sigma^2\,S^{(4)} + \sigma^{\frac{5}{2}}\,S^{(5)} + \sigma^3\,S^{(6)} + \sigma^{\frac{7}{2}}\,S^{(7)} + \sigma^4\,S^{(8)} && (2.20\text{a})\\
\Delta^{\frac{1}{2}} &\sim 1 + \sigma\,\Delta^{(2)} + \sigma^{\frac{3}{2}}\,\Delta^{(3)} + \sigma^2\,\Delta^{(4)} + \sigma^{\frac{5}{2}}\,\Delta^{(5)} + \sigma^3\,\Delta^{(6)} && (2.20\text{b})\\
\bar{\Delta}^{\frac{1}{2}} &\sim 1 + \sigma\,\bar{\Delta}^{(2)} + \sigma^{\frac{3}{2}}\,\bar{\Delta}^{(3)} + \sigma^2\,\bar{\Delta}^{(4)} + \sigma^{\frac{5}{2}}\,\bar{\Delta}^{(5)} + \sigma^3\,\bar{\Delta}^{(6)}. && (2.20\text{c})
\end{aligned}$$

(Recall the scalar expansion coefficients are related to the tensor expansion cofficients via $A^{(n)} = 2^{\frac{n}{2}}\,p^{p_1}\cdots p^{p_n}\,A^{(n)}_{p_1\cdots p_n}$, where in general we take $\sqrt{\sigma}\,p^a = \sigma^{;a}$).

Putting these in (2.10), the expansion coefficients $G^{(n)}_{\rm div,ren}$ are

$$\begin{aligned}
G^{(0)}_{\rm div,ren} &= \bar{\Delta}^{(2)} - \Delta^{(2)} - S^{(4)} && (2.21\text{a})\\
G^{(1)}_{\rm div,ren} &= \bar{\Delta}^{(3)} - \Delta^{(3)} - S^{(5)} && (2.21\text{b})\\
G^{(2)}_{\rm div,ren} &= \bar{\Delta}^{(4)} - \Delta^{(4)} - \Delta^{(2)}\,S^{(4)} + S^{(4)}\left(-\bar{\Delta}^{(2)} + \Delta^{(2)} + S^{(4)}\right) - S^{(6)} && (2.21\text{c})\\
G^{(3)}_{\rm div,ren} &= \bar{\Delta}^{(5)} - \Delta^{(5)} - \Delta^{(3)}\,S^{(4)} - \Delta^{(2)}\,S^{(5)} + \left(-\bar{\Delta}^{(2)} + \Delta^{(2)} + S^{(4)}\right)\,S^{(5)}\\
&\quad + S^{(4)}\left(-\bar{\Delta}^{(3)} + \Delta^{(3)} + S^{(5)}\right) - S^{(7)} && (2.21\text{d})\\
G^{(4)}_{\rm div,ren} &= \bar{\Delta}^{(6)} - \Delta^{(6)} - \Delta^{(4)}\,S^{(4)} - \Delta^{(3)}\,S^{(5)} + S^{(5)}\left(-\bar{\Delta}^{(3)} + \Delta^{(3)} + S^{(5)}\right)\\
&\quad - \left(\left(-\bar{\Delta}^{(2)} + \Delta^{(2)} + S^{(4)}\right)\left({S^{(4)}}^2 - S^{(6)}\right)\right) - \Delta^{(2)}\,S^{(6)}\\
&\quad + S^{(4)}\left(-\bar{\Delta}^{(4)} + \Delta^{(4)} + \Delta^{(2)}\,S^{(4)} + S^{(6)}\right) - S^{(8)} && (2.21\text{e})
\end{aligned}$$

We have shown that for $G^{(0)}_{\rm div,ren}$ a direction dependence could arise, but when we use the values of the expansion tensors for Σ, $\Delta^{\frac{1}{2}}$ and $\bar{\Delta}^{\frac{1}{2}}$, this direction dependence cancels. We expect this to happen because we see the coefficients that contribute to any given coefficient of $G_{\rm div,ren}$ are of a higher power than the $G^{(n)}_{\rm div,ren}$ coefficient in question. When going from the scalar expansion coefficient form to the tensor expansion form, the rank of the tensor is the same as the order of the coefficient. And each of the $G^{(n)}_{\rm div,ren}$ is made up of higher order coefficients. But as we find by direct substitution from the expansion tensors listed in Appendix A, for each case the direction dependence always cancel.

In Appendix B, we give the complete form of the coefficients $G^{(n)}_{\text{div,ren}}$, in their corresponding tensorial form. This is one of our main results: The regularization of the leading order divergence of the Green function, when Green function has been computed in an optical metric conformal to the physical metric in which the problem is given. What is new here is that this regularization has been carried out to the order needed for the computation of the noise kernel. This analysis has been developed so as to take advantage of the symbolic computing potential of current workstations.

The last remaining obstacle in computing the noise kernel comes from G_{fin}. As it stands, it is defined in terms of the optical metric, while it is the physical metric with which we need to take the covariant derivatives that determine the noise kernel. By using the function Σ and the series expansions (2.11b), (2.11c) and (2.11c), along with the series expansions

$$(\delta\tau)^2 = \sigma\,\delta\tau^{(2)} + \sigma^{\frac{3}{2}}\,\delta\tau^{(3)} + \sigma^2\,\delta\tau^{(4)} \tag{2.22}$$

$$e^{-a(\omega+\omega')} = \omega^{(0,a)} + \sqrt{\sigma}\,\omega^{(1,a)} + \sigma\,\omega^{(2,a)} + \sigma^{\frac{3}{2}}\,\omega^{(3,a)} + \sigma^2\,\omega^{(4,a)} \tag{2.23}$$

G_{fin} can be expanded in terms of the physical σ as

$$G_{\text{fin}} = \frac{1}{8\,\pi^2}\left(G^{(0)}_{\text{fin}} + \sqrt{\sigma}\,G^{(1)}_{\text{fin}} + \sigma\,G^{(2)}_{\text{fin}} + \sigma^{\frac{3}{2}}\,G^{(3)}_{\text{fin}} + \sigma^2\,G^{(4)}_{\text{fin}}\right) \tag{2.24}$$

where these expansion coefficients are

$$G^{(0)}_{\text{fin}} = \frac{\kappa^2}{6}\omega^{(0,1)} \tag{2.25a}$$

$$G^{(1)}_{\text{fin}} = \frac{\kappa^2}{6}\omega^{(1,1)} \tag{2.25b}$$

$$G^{(2)}_{\text{fin}} = \frac{\kappa^2}{6}\left(\bar{\Delta}^{(2)}\,\omega^{(0,1)} + \omega^{(2,1)}\right) + \frac{\kappa^4}{180}\left(2\,\delta\tau^{(2)}\,\omega^{(0,1)} - \omega^{(0,2)}\right) \tag{2.25c}$$

$$\begin{aligned} G^{(3)}_{\text{fin}} = {} & \frac{\kappa^2}{6}\left(\bar{\Delta}^{(3)}\,\omega^{(0,1)} + \bar{\Delta}^{(2)}\,\omega^{(1,1)} + \omega^{(3,1)}\right) \\ & + \frac{\kappa^4}{180}\left(2\,\delta\tau^{(3)}\,\omega^{(0,1)} + 2\,\delta\tau^{(2)}\,\omega^{(1,1)} - \omega^{(1,2)}\right) \end{aligned} \tag{2.25d}$$

$$\begin{aligned} G^{(4)}_{\text{fin}} = {} & \frac{\kappa^2}{6}\left(\bar{\Delta}^{(4)}\,\omega^{(0,1)} + \bar{\Delta}^{(3)}\,\omega^{(1,1)} + \bar{\Delta}^{(2)}\,\omega^{(2,1)} + \omega^{(4,1)}\right) \\ & - \frac{\kappa^4}{180}\Big(-2\,\delta\tau^{(4)}\,\omega^{(0,1)} + S^{(4)}\,\omega^{(0,2)} + \bar{\Delta}^{(2)}\,\left(-2\,\delta\tau^{(2)}\,\omega^{(0,1)} + \omega^{(0,2)}\right) \\ & \qquad -2\,\delta\tau^{(3)}\,\omega^{(1,1)} - 2\,\delta\tau^{(2)}\,\omega^{(2,1)} + \omega^{(2,2)}\Big) \\ & + \frac{\kappa^6}{3780}\left(4\,{\delta\tau^{(2)}}^2\,\omega^{(0,1)} - 6\,\delta\tau^{(2)}\,\omega^{(0,2)} + \omega^{(0,3)}\right) \end{aligned} \tag{2.25e}$$

The explicit expressions for these expansion tensors are given in Appendix C.

With this we have regularized and expanded $G_{\text{div,ren}}$, and expanded in the physical metric G_{fin}, both to fourth order. The Hadamard ansatz function W's series expansion is derived in Appendix E of Paper II, also to fourth order. From here we proceed as we did in Paper II for the optical metrics. To review, once a metric is selected, the component values of the expansions of $G_{\text{div,ren}}$, G_{fin} and W are computed symbolically on the computer. Then Eqn (3.10) of Paper II is used to determine the component values of the coincident limit of up to four covariant derivatives of G_{ren}, along with the needed covariant derivatives of the coincident limits. From here the component values of the coincident limit of the noise kernel are computed. This procedure is adopted owing to the large size of the expansions. In fact, initial attempts to delay the specification of the metric until a full determination of the noise kernel ended up yielding a general tensorial expression of nearly 60,000 terms. By working out from the expansion tensors in terms of their component values, we found this had the added advantage of enabling one to study each of the separate terms that go into computing the noise kernel.

III. SCHWARZSCHILD BLACK HOLE

We can now turn our attention to a specific example. Consider a massless, conformally coupled scalar field on a Schwarzschild black hole with mass M. The line element is given in the usual coordinates $x^a = (r, \theta, \phi, \tau)$

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\!\theta\,d\phi^2\right) \tag{3.1}$$

where τ is the imaginary time in the Euclidean sector of the spacetime. With the conformal factor

$$e^{2w} = 1 - \frac{2M}{r} \tag{3.2}$$

this metric is the physical metric corresponding to the optical metric considered in Paper II. As in that case, the imaginary time dimension has periodicity $\kappa = 1/4M$, corresponding to a temperature associated with the Hartle-Hawking state. We use the scaled spatial coordinate $x \equiv 2M/r \equiv 1/2\kappa r$. With this choice, the black hole horizon $r = 2M$ is at $x = 1$ while spatial infinity is at $x = 0$.

We now use the results for the expansion tensors above to compute the noise kernel coincident limit, along with the stress tensor itself. For the stress tensor:

$$\begin{aligned}\left\langle T_a{}^b \right\rangle = (\rho_\infty)\,\mathrm{diag}\,\{ \quad & \left(1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4 + 6\,x^5 + 15\,x^6\right)/3, \\ & \left(1 + 2\,x + 3\,x^2\right)\,\left(1 + 4\,x^3 - 3\,x^4\right)/3, \\ & \left(1 + 2\,x + 3\,x^2\right)\,\left(1 + 4\,x^3 - 3\,x^4\right)/3, \\ & -1 - 2\,x - 3\,x^2 - 4\,x^3 - 5\,x^4 - 6\,x^5 + 33\,x^6 \ \} \end{aligned} \tag{3.3}$$

where

$$\rho_\infty = \frac{\kappa^4}{480\,\pi^2} = -\left.\left\langle T_\tau{}^\tau \right\rangle\right|_{r\to\infty} \tag{3.4}$$

This result agrees with Page's Eq. (83) [3]. In his work, Page showed the stress tensor must satisfy a functional-differential scale equation under conformal transformations. He then found, via trial and error, a general solution to this equation. This became the basis for his computation of the stress tensor in the black hole metric.

In contrast, we have worked directly with the Green function and the point separation definition of the stress tensor. The conformal transformation properties of the geometric objects that go into the Green function are studied and the corresponding series expansions are developed. As can be seen from Appendices B and C, these expansions can get quite involved even for just up to the second order terms needed for the stress tensor. Thus our agreement with Page on the stress tensor serves as an anchor for this work. Moreover, the methods used for determining the series expansions are fundamentally recursive. The zero order result (2.18) and the second order result (3.3) are in agreement with known results. Since the terms needed for the noise kernel are generated recursively from these the symbolically computed results for the Schwarzschild noise kernel coincident limit should be accurate, up to the validity of the Gaussian approximation to the Green function. These results are

$$\begin{aligned} N_\tau{}^\tau{}_\tau{}^\tau \;=\; & \frac{{\rho_\infty}^2}{756}\left(219 + 876\,x + 2190\,x^2 + 4380\,x^3 + 7215\,x^4 + 10464\,x^5 + \right. \\ & +16920\,x^6 + 21424\,x^7 - 2984943\,x^8 + 219140\,x^9 + 197314\,x^{10} \\ & \left. +180260\,x^{11} + 3292493\,x^{12}\right) \end{aligned} \tag{3.5a}$$

$$\begin{aligned} N_r{}^r{}_r{}^r \;=\; & \frac{{\rho_\infty}^2}{2268}\left(137 + 548\,x + 1370\,x^2 + 3860\,x^3 + 9005\,x^4 + 98432\,x^5 \right. \\ & +225080\,x^6 + 408752\,x^7 + 2553371\,x^8 + 1725900\,x^9 + 1206822\,x^{10} \\ & \left. +761100\,x^{11} - 4090521\,x^{12}\right) \end{aligned} \tag{3.5b}$$

$$\begin{aligned} N_\theta{}^\theta{}_\theta{}^\theta \;=\; & \frac{{\rho_\infty}^2}{2268}\left(137 + 548\,x + 1370\,x^2 + +2180\,x^3 + 3965\,x^4 + 37952\,x^5 \right. \\ & +77240\,x^6 + 102992\,x^7 - 11973349\,x^8 + 1817100\,x^9 + 1721382\,x^{10} \\ & \left. +1630140\,x^{11} + 12756759\,x^{12}\right) \end{aligned} \tag{3.5c}$$

$$\begin{aligned} N_\tau{}^\tau{}_r{}^r \;=\; & \frac{{\rho_\infty}^2}{2268}\left(-219 - 876\,x - 2190\,x^2 - 2140\,x^3 - 495\,x^4 + 2976\,x^5 \right. \\ & -10984\,x^6 - 49872\,x^7 + 1327551\,x^8 - 230916\,x^9 - 50834\,x^{10} \\ & \left. +95356\,x^{11} - 774845\,x^{12}\right) \end{aligned} \tag{3.5d}$$

$$\begin{aligned} N_\tau{}^\tau{}_\theta{}^\theta \;=\; & \frac{{\rho_\infty}^2}{2268}\left(-219 - 876\,x - 2190\,x^2 - 5500\,x^3 - 10575\,x^4 - 17184\,x^5 \right. \\ & -19888\,x^6 - 7200\,x^7 - 10917561\,x^8 + 1056828\,x^9 + 999526\,x^{10} \\ & \left. +952012\,x^{11} + 11087083\,x^{12}\right) \end{aligned} \tag{3.5e}$$

$$\begin{aligned} N_r{}^r{}_\theta{}^\theta \;=\; & \frac{{\rho_\infty}^2}{2268}\left(41 + 164\,x + 410\,x^2 - 860\,x^3 - 4255\,x^4 - 50704\,x^5 \right. \\ & -107048\,x^6 - 179440\,x^7 + 1023059\,x^8 + 159708\,x^9 + 329206\,x^{10} \end{aligned}$$

$$+478972\,x^{11}+1465003\,x^{12}\Big) \tag{3.5f}$$

Knowing the component values, we determine the trace to be

$$N_p{}^p{}_q{}^q = \frac{32000\,\rho_\infty{}^2\,x^8}{21}\,(1+x)\,(-27+31\,x)\,\left(1+x^2\right) \tag{3.6}$$

As with the optical Schwarzschild case, the trace under the Gaussian approximation fails to vanish, which it should, for the massless conformal coupling case we are considering. By taking the trace of the coincident limit expressions for the noise kernel above, term by term, we find that this arises from the non-vanishing of the fourth order derivative terms such as $[G_{\mathrm{ren};p}{}^p{}_q{}^q]$. This is what we had expected, based on our analysis of the optical Schwarzschild metric undertaken in Paper II. It is not from our implementation of the conformal transformation.

Our result yields a zero trace at spatial infinity $x \to 0$, or $r \to \infty$. Using the fluctuation measure

$$\Delta_{abcd} = \left|\frac{\langle T_{ab}T_{cd}\rangle - \langle T_{ab}\rangle\,\langle T_{cd}\rangle}{\langle T_{ab}T_{cd}\rangle}\right| = \left|\frac{4N_{abcd}}{4N_{abcd}+\langle T_{ab}\rangle\,\langle T_{cd}\rangle}\right|, \tag{3.7}$$

at $r \to \infty$, we get

$$\begin{array}{rcccccc} abcd: & \tau\tau\tau\tau & rrrr & \theta\theta\theta\theta & \tau\tau rr & \tau\tau\theta\theta & rr\theta\theta \\ \Delta_{abcd}: & \frac{73}{136} & \frac{137}{200} & \frac{137}{200} & \frac{73}{136} & \frac{73}{136} & \frac{41}{104} \end{array} \tag{3.8}$$

As we can expect for a Hartle-Hawking state, this matches the thermal results of hot flat space we obtained in Paper II.

We also report the magnitude of the error at the horizon:

$$\begin{array}{rcccccc} abcd: & \tau\tau\tau\tau & rrrr & \theta\theta\theta\theta & \tau\tau rr & \tau\tau\theta\theta & rr\theta\theta \\ \frac{N_p{}^p{}_q{}^q}{N_a{}^b{}_c{}^d}: & 1904\% & 1904\% & 894\% & 18278\% & 1775\% & 1775\% \end{array} \tag{3.9}$$

These results show the Gaussian approximation has completely broken down at the horizon.

This is not to say we can draw no conclusions other than to discover the inadequacies of the Gaussian approximation. What is important is the finiteness at the horizon of both the noise kernel components and the error as expressed by the failure of the trace of the noise kernel to vanish. In our computation of both the stress tensor and the noise kernel, we have discovered finiteness at the horizon is a "fragile" property. By this we mean any small error in the symbolic code would result in a noise kernel that diverges as $x \to 1$. This has lead us to develop more than one way to determine the series for the optical metric geometric objects, just to test the symbolic code. We arrive at the results (3.6) using more than one computational path. In contrast if there were one single error in the code, the resulting noise kernel cannot be finite on the horizon.

It is the noise kernel itself, via its trace, that provides a measure of the error of the Gaussian approximation, *i.e.*, we self-consistently compute both the noise kernel and its error. Statements that address correcting the error of the Gaussian approximation also apply to the noise kernel. Correcting the Gaussian approximation will amount to finding the terms that need to be included such that it satisfies the field equation to fourth order in σ^a. When this corrected approximation is then in turn used to compute the noise kernel, it will have to correct the current noise kernel results (3.5f) in such a manner as to exactly cancel the current trace (3.6). Hence the correction to the noise kernel will itself be finite at the horizon. With this in mind, we can conclude the fluctuations of the stress tensor, as measured by the coincident limit of the noise kernel, are finite at the black-hole's horizon. This is one of the main lessons learned from the analysis of the noise kernel, as derived via point separation.

IV. DISCUSSIONS

Let us summarize our findings pertaining to two sets of issues: the range of validity of the Gaussian approximation and the results and usefulness of our program in spite of this approximation.

Despite its success for the stress tensor calculation there is no compelling reason to expect the Gaussian approximated Green function to produce reasonable results for expressions involving higher order covariant derivatives, as in the noise kernel. Nonetheless the Gaussian approximation is very relevant because it contains the leading order divergence. This structure will remain even with a better approximation, while this leading order divergence must be regularized. This step is needed regardless of what form of the Green function one adopts – future improvements to the Gaussian approximation remains desirable, or, if the exact Green function is derived in the optical metric only. Our work lays down the structure and provides the details for its implementation.

Now for the successes and failures of the Gaussian approximation as applied to the computation of the noise kernel. On the positive side our results for the fluctuations of the stress tensor of the Hawking flux in the far field region checks with the analytic results of Campos and Hu [2]. A fringe benefit is that we can verify our procedure by explicitly re-deriving the Page result [3] for the stress tensor. We note that in Page's original work, the direct use of the conformal transformation was circumvented by "guessing" the solution to a functional differential equation. Our result is the first we know where the methodology of point separation was carried all the way through to the final result. That we get the known results is a check on our method and its correct implementation. On the negative side, our calculation shows that the fluctuations of the stress tensor based on the Gaussian approximation is unreliable in regions close to the event horizon. We show this by checking that the trace anomaly fails to vanish there. This result is not unexpected, as can be inferred from our findings in Paper II. Corrections to the Gaussian approximation need be introduced to improve the accuracy. One important result which may be under-appreciated is that we find a *finite* expression for the noise kernel. Even though the error is large, as long as it is finite, we know corrections to the approximations will themselves be finite. Hence, where the approximate noise kernel is finite, we can expect the full noise kernel to be finite. That the noise kernel on the horizon of a Schwarzschild black hole is finite is in itself a qualitatively significant result. This dispels claims to the contrary based on intuitive arguments or less rigorous calculations [5, 6, 7, 8].

Along the way to regularizing the Green function to fourth order, necessary for the coincident limit of the noise kernel, we have developed the series expansions of the various geometric objects that make up the Green function to a high order. Once the additional terms necessary to correct the Gaussian approximation are determined, it is a simple matter to use the work contained herein to compute these correction terms to a high enough order. In this sense, this work not only lays down the tracks and defines the steps, but also provides all the details necessary for implementing the point separation program to calculate the regularized noise kernel for quantum fields in curved spacetime. To end with a more practical note, for black hole fluctuations and backreaction calculations [9, 10] which requires results from investigations like ours here, we can only reiterate what other researchers have said (lamented) before: A better approximation to the Green function is to be desired.

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APPENDIX A: CONFORMAL TRANSFORMATION

Since we compute the Green function in the optical metric conformally related to the physical metric we are interested in, we need to know how the geometric objects we use conformally transform. We denote objects in the optical metric with an overbar and the covariant derivative commensurate with the optical metric by a vertical stroke. The two metrics are related via

$$\bar{g}_{ab} = e^{-2\omega} g_{ab} \tag{A1}$$

We develop the series expansions of the world function $\bar{\sigma}$, to eighth order, and $\bar{\Delta}^{1/2}$ to sixth order. We also need the function

$$\Sigma(x,y) = e^{\omega(x)+\omega(y)}\bar{\sigma} - \sigma, \tag{A2}$$

whose series expansion readily follows once the series for $\bar{\sigma}$ is determined. The most straightforward method for determining these series expansions is to start by considering the conformal transformation of the differential equations each must satisfy.

Considering first $\bar{\sigma}$, it satisfies, in terms of the optical metric, the equation

$$\bar{\sigma} = \frac{1}{2}\bar{\sigma}_{|p}\bar{\sigma}^{|p} = \frac{1}{2}\bar{g}^{pq}\bar{\sigma}_{|p}\bar{\sigma}_{|q}. \tag{A3}$$

Using $\bar{\sigma}_{|a} = \bar{\sigma}_{,a} = \bar{\sigma}_{;a}$, this equation transforms to

$$\bar{\sigma} = \frac{e^{2\omega}}{2} g^{pq}\bar{\sigma}_{;p}\bar{\sigma}_{;q} = \frac{e^{2\omega}}{2}\bar{\sigma}_{;p}\bar{\sigma}^{;p} \tag{A4}$$

and we now have an equation purely in terms of the physical metric. We still have

$$[\bar{\sigma}] = 0, \qquad\qquad [\bar{\sigma}_{;a}] = 0. \tag{A5}$$

and we assume the series expansion

$$\begin{aligned}\bar{\sigma} \;=\; & \Upsilon^{(2)}_{pq}\sigma^p\sigma^q + \Upsilon^{(3)}_{pqr}\sigma^p\sigma^q\sigma^r + \Upsilon^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s + \Upsilon^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t \\ & +\Upsilon^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u + \Upsilon^{(7)}_{pqrstuv}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v \\ & +\Upsilon^{(8)}_{pqrstuvw}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v\sigma^w \end{aligned} \tag{A6}$$

To determine the expansion tensors, we proceed as we did in Paper II for $\Delta^{1/2}$ and substitute the expansion into the differential equation (A4) and collect terms by their order in σ^a.

The order σ term must satisfy

$$2\,\Upsilon^{(2)}{}_{ab} - e^{2\,\omega}\,\left(\Upsilon^{(2)}{}_{ap}\,\Upsilon^{(2)}{}_{b}{}^{p} + \Upsilon^{(2)}{}_{pa}\,\Upsilon^{(2)p}{}_{b} + 2\,\Upsilon^{(2)}{}_{ap}\,\Upsilon^{(2)p}{}_{b}\right) = 0. \tag{A7}$$

This has the solution

$$\Upsilon^{(2)}{}_{ab} = \frac{g_{ab}}{2\,e^{2\,\omega}}, \tag{A8a}$$

which can be seen via substitution. Since $\bar{\sigma}$ is a symmetric function, we use the results of Paper II's Appendix B, which give the odd order expansion coefficients in terms of the even order one. Thus the third order coefficient is

$$\Upsilon^{(3)}{}_{abc} = -\frac{\Upsilon^{(2)}{}_{ab;c}}{2} = \frac{\omega_{;c}\,g_{ab}}{2\,e^{2\,\omega}} \tag{A8b}$$

With this, we have set up the recursion. Now it is just a matter to proceed as with the computation of the series expansion Van Vleck-Morette determinant as presented in Paper II. The results for the rest of the expansion tensors for $\bar{\sigma}$ are:

$$e^{2\,\omega}\,\Upsilon^{(4)}{}_{abcd} \;\doteq\; \frac{g_{cd}}{6}\,(2\,\omega_{;a}\,\omega_{;b} - \omega_{;ab}) - \frac{g_{ab}\,g_{cd}}{24}\omega_{;p}\,\omega^{;p} \tag{A8c}$$

$$\begin{aligned} e^{2\,\omega}\,\Upsilon^{(5)}{}_{abcde} \;\doteq\; & \frac{g_{ab}}{24}\,(4\,\omega_{;c}\,\omega_{;d}\,\omega_{;e} - 6\,\omega_{;c}\,\omega_{;de} + \omega_{;cde}) \\ & -\frac{g_{ab}\,g_{cd}}{24}\omega_{;p}\,\left(\omega_{;e}\,\omega^{;p} - \omega_{;e}{}^{p}\right) \end{aligned} \tag{A8d}$$

$$\begin{aligned} e^{2\,\omega}\,\Upsilon^{(6)}{}_{abcdef} \;\doteq\; & \frac{g_{ab}}{120}\,(8\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f} - 24\,\omega_{;c}\,\omega_{;d}\,\omega_{;ef} + 6\,\omega_{;cd}\,\omega_{;ef} \\ & \quad +8\,\omega_{;c}\,\omega_{;def} - \omega_{;cdef}) \\ & -\frac{g_{ab}\,g_{cd}}{720}\,(18\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega^{;p} - 8\,\omega_{;p}\,\omega^{;p}\,\omega_{;ef} \\ & \quad +9\,\omega_{;p}\,\omega_{;ef}{}^{p} + 12\,\omega_{;p}\,\omega_{;q}\,R_{e}{}^{q}{}_{f}{}^{p}) \\ & +\frac{1}{720}g_{ab}\,g_{cd}\,g_{ef}\omega_{;p}\,\omega_{;q}\,\left(\omega^{;p}\,\omega^{;q} - 3\,\omega^{;pq}\right) \end{aligned} \tag{A8e}$$

$$\begin{aligned} e^{2\,\omega}\,\Upsilon^{(7)}{}_{abcdefg} \;\doteq\; & \frac{g_{ab}}{720}\,(16\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g} - 80\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;fg} \\ & \quad +60\,\omega_{;c}\,\omega_{;de}\,\omega_{;fg} + 40\,\omega_{;c}\,\omega_{;d}\,\omega_{;efg} - 20\,\omega_{;cd}\,\omega_{;efg} \\ & \quad -10\,\omega_{;c}\,\omega_{;defg} + \omega_{;cdefg}) \\ & -\frac{g_{ab}\,g_{cd}}{4320}\,(48\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega^{;p} - 27\,\omega_{;p}\,\omega_{;q}\,R_{e}{}^{q}{}_{f}{}^{p}{}_{;g} \\ & \quad -66\,\omega_{;p}\,\omega_{;e}\,\omega^{;p}\,\omega_{;fg} + 50\,\omega_{;p}\,\omega_{;e}{}^{p}\,\omega_{;fg} \\ & \quad -156\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}{}^{p} + 72\,\omega_{;e}\,\omega_{;pf}\,\omega_{;g}{}^{p} + 22\,\omega_{;p}\,\omega_{;ef}\,\omega_{;g}{}^{p} \\ & \quad +9\,\omega_{;p}\,\omega^{;p}\,\omega_{;efg} + 78\,\omega_{;p}\,\omega_{;e}\,\omega_{;fg}{}^{p} - 24\,\omega_{;pe}\,\omega_{;fg}{}^{p} \\ & \quad -6\,\omega_{;pe}\,\omega_{;f}{}^{p}{}_{g} - 12\,\omega_{;p}\,\omega_{;efg}{}^{p} - 6\,\omega_{;p}\,\omega_{;ef}{}^{p}{}_{g} \\ & \quad 6\,\omega_{;p}\,\omega_{;e}{}^{p}{}_{fg} - 32\,\omega_{;p}\,\omega_{;qe}\,R_{f}{}^{p}{}_{g}{}^{q} + 96\,\omega_{;p}\,\omega_{;q}\,\omega_{;e}\,R_{f}{}^{q}{}_{g}{}^{p} \\ & \quad -52\,\omega_{;p}\,\omega_{;qe}\,R_{f}{}^{q}{}_{g}{}^{p}) \\ & +\frac{1}{8640}g_{ab}\,g_{cd}\,g_{ef}\omega_{;p}\,(12\,\omega_{;q}\,\omega_{;g}\,\omega^{;p}\,\omega^{;q} - 14\,\omega_{;q}\,\omega^{;q}\,\omega_{;g}{}^{p} \\ & \quad -10\,\omega_{;q}\,\omega^{;p}\,\omega_{;g}{}^{q} + 9\,\omega_{;q}{}^{p}\,\omega_{;g}{}^{q} - 36\,\omega_{;q}\,\omega_{;g}\,\omega^{;pq} \\ & \quad +27\,\omega_{;qg}\,\omega^{;pq} - 6\,\omega_{;q}\,\omega_{;g}{}^{pq} + 18\,\omega_{;q}\,\omega_{;g}{}^{qp} \end{aligned}$$

$$+6\,\omega_{;q}\,\omega^{;pq}{}_{g} - 6\,\omega_{;q}\,\omega_{;r}\,R_g{}^{pqr}) \tag{A8f}$$

$$\begin{aligned}
e^{2\,\omega}\,\Upsilon^{(8)}{}_{abcdefgh} \;\doteq\;& \frac{g_{ab}}{5040}\,(32\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega_{;h} - 240\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}\,\omega_{;gh} \\
&\quad +360\,\omega_{;c}\,\omega_{;d}\,\omega_{;ef}\,\omega_{;gh} - 60\,\omega_{;cd}\,\omega_{;ef}\,\omega_{;gh} \\
&\quad +160\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;fgh} - 240\,\omega_{;c}\,\omega_{;de}\,\omega_{;fgh} \\
&\quad +20\,\omega_{;cde}\,\omega_{;fgh} - 60\,\omega_{;c}\,\omega_{;d}\,\omega_{;efgh} + 30\,\omega_{;cd}\,\omega_{;efgh} \\
&\quad +12\,\omega_{;c}\,\omega_{;defgh} - \omega_{;cdefgh}) \\
& -\frac{g_{ab}\,g_{cd}}{120960}\,(480\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega_{;h}\,\omega^{;p} \\
&\quad -1360\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega^{;p}\,\omega_{;gh} + 308\,\omega_{;p}\,\omega^{;p}\,\omega_{;ef}\,\omega_{;gh} \\
&\quad +1936\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}{}^{p}\,\omega_{;gh} - 200\,\omega_{;pe}\,\omega_{;f}{}^{p}\,\omega_{;gh} \\
&\quad -2080\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega_{;h}{}^{p} + 1552\,\omega_{;e}\,\omega_{;f}\,\omega_{;pg}\,\omega_{;h}{}^{p} \\
&\quad +1048\,\omega_{;p}\,\omega_{;e}\,\omega_{;fg}\,\omega_{;h}{}^{p} - 504\,\omega_{;pe}\,\omega_{;fg}\,\omega_{;h}{}^{p} \\
&\quad +384\,\omega_{;p}\,\omega_{;e}\,\omega^{;p}\,\omega_{;fgh} - 444\,\omega_{;p}\,\omega_{;e}{}^{p}\,\omega_{;fgh} \\
&\quad +1584\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;gh}{}^{p} - 1128\,\omega_{;e}\,\omega_{;pf}\,\omega_{;gh}{}^{p} \\
&\quad -828\,\omega_{;p}\,\omega_{;ef}\,\omega_{;gh}{}^{p} + 6\,\omega_{;pef}\,\omega_{;gh}{}^{p} \\
&\quad +135\,\omega_{;efp}\,\omega_{;gh}{}^{p} + 48\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}{}^{p}{}_{h} \\
&\quad -264\,\omega_{;e}\,\omega_{;pf}\,\omega_{;g}{}^{p}{}_{h} + 72\,\omega_{;p}\,\omega_{;ef}\,\omega_{;g}{}^{p}{}_{h} \\
&\quad +84\,\omega_{;pef}\,\omega_{;g}{}^{p}{}_{h} - 90\,\omega_{;efp}\,\omega_{;g}{}^{p}{}_{h} - 36\,\omega_{;p}\,\omega^{;p}\,\omega_{;efgh} \\
&\quad -504\,\omega_{;p}\,\omega_{;e}\,\omega_{;fgh}{}^{p} + 144\,\omega_{;pe}\,\omega_{;fgh}{}^{p} \\
&\quad -288\,\omega_{;p}\,\omega_{;e}\,\omega_{;fg}{}^{p}{}_{h} + 96\,\omega_{;pe}\,\omega_{;fg}{}^{p}{}_{h} \\
&\quad +264\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}{}^{p}{}_{gh} - 48\,\omega_{;pe}\,\omega_{;f}{}^{p}{}_{gh} + 60\,\omega_{;p}\,\omega_{;efgh}{}^{p} \\
&\quad +36\,\omega_{;p}\,\omega_{;efg}{}^{p}{}_{h} + 12\,\omega_{;p}\,\omega_{;ef}{}^{p}{}_{gh} - 48\,\omega_{;p}\,\omega_{;e}{}^{p}{}_{fgh} \\
&\quad +36\,\omega_{;p}\,\omega_{;efg}{}^{p}{}_{h} + 12\,\omega_{;p}\,\omega_{;ef}{}^{p}{}_{gh} - 48\,\omega_{;p}\,\omega_{;e}{}^{p}{}_{fgh} \\
&\quad -1068\,\omega_{;p}\,\omega_{;q}\,\omega_{;e}\,R_f{}^{q}{}_{g}{}^{p}{}_{;h} - 912\,\omega_{;p}\,R_{qef}{}^{p}{}_{;g}\,\omega_{;h}{}^{q} \\
&\quad +180\,\omega_{;p}\,\omega_{;q}\,R_e{}^{q}{}_{f}{}^{p}{}_{;gh} - 336\,\omega_{;p}\,\omega_{;q}\,R_{ref}{}^{p}\,R_g{}^{q}{}_{h}{}^{r} \\
&\quad -8\,\omega_{;p}\,R_g{}^{p}{}_{h}{}^{q}\,(171\,\omega_{;e}\,\omega_{;qf} + 14\,\omega_{;qef} - 45\,\omega_{;efq}) \\
&\quad +8\,R_g{}^{q}{}_{h}{}^{p}\,(232\,\omega_{;p}\,\omega_{;q}\,\omega_{;e}\,\omega_{;f} - 271\,\omega_{;p}\,\omega_{;e}\,\omega_{;qf} \\
&\qquad\qquad +82\,\omega_{;pe}\,\omega_{;qf} - 120\,\omega_{;p}\,\omega_{;q}\,\omega_{;ef} \\
&\qquad\qquad +56\,\omega_{;p}\,\omega_{;qef} + 15\,\omega_{;p}\,\omega_{;efq})) \\
& \frac{g_{ab}\,g_{cd}\,g_{ef}}{241920}\,(192\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega_{;h}\,\omega^{;p}\,\omega^{;q} \\
&\quad -96\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}\,\omega_{;gh} - 360\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega^{;q}\,\omega_{;h}{}^{p} \\
&\quad +464\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}{}^{q}\,\omega_{;h}{}^{p} - 408\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega^{;p}\,\omega_{;h}{}^{q} \\
&\quad 160\,\omega_{;p}\,\omega^{;p}\,\omega_{;qg}\,\omega_{;h}{}^{q} + 316\,\omega_{;p}\,\omega_{;g}\,\omega_{;q}{}^{p}\,\omega_{;h}{}^{q} \\
&\quad + - 64\,\omega_{;pg}\,\omega_{;q}{}^{p}\,\omega_{;h}{}^{q} - 64\,\omega_{;pq}\,\omega_{;g}{}^{p}\,\omega_{;h}{}^{q} \\
&\quad -544\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega_{;h}\,\omega^{;pq} + 948\,\omega_{;p}\,\omega_{;g}\,\omega_{;qh}\,\omega^{;pq} \\
&\quad -128\,\omega_{;pg}\,\omega_{;qh}\,\omega^{;pq} + 216\,\omega_{;p}\,\omega_{;q}\,\omega_{;gh}\,\omega^{;pq} \\
&\quad +168\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;gh}{}^{p} - 144\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;gh}{}^{q} \\
&\quad -300\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;gh}{}^{q} - 40\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;g}{}^{p}{}_{h} \\
&\quad +184\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;g}{}^{q}{}_{h} - 288\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega_{;h}{}^{pq} \\
&\quad -96\,\omega_{;p}\,\omega_{;qg}\,\omega_{;h}{}^{pq} + 672\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega_{;h}{}^{qp} \\
&\quad -264\,\omega_{;p}\,\omega_{;qg}\,\omega_{;h}{}^{qp} + 240\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega^{;pq}{}_{h} \\
&\quad -168\,\omega_{;p}\,\omega_{;qg}\,\omega^{;pq}{}_{h} - 12\,\omega_{;p}\,\omega_{;q}\,\omega_{;gh}{}^{pq} \\
&\quad -132\,\omega_{;p}\,\omega_{;q}\,\omega_{;gh}{}^{qp} + 66\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}{}^{p}{}_{h}{}^{q} \\
&\quad +54\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}{}^{pq}{}_{h} + 6\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}{}^{q}{}_{h}{}^{p} \\
&\quad -78\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}{}^{qp}{}_{h} - 48\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}{}_{gh} \\
&\quad -3\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,(15\,R_g{}^{p}{}_{h}{}^{q;r} - 13\,R_g{}^{pqr}{}_{;h} \\
&\qquad\qquad +30\,R_g{}^{r}{}_{h}{}^{p;q} + 15\,R_g{}^{r}{}_{h}{}^{q;p} + 13\,R_g{}^{rpq}{}_{;h}) \\
&\quad + - 80\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{q}\,(5\,R_g{}^{p}{}_{h}{}^{r} + 4\,R_g{}^{r}{}_{h}{}^{p}) \\
&\quad +48\,\omega_{;p}\,\omega_{;q}\,(3\,\omega_{;rg}\,R_h{}^{pqr} + 5\,\omega_{;r}\,\omega_{;g}\,R_h{}^{qpr}) \\
&\quad 32\,\omega_{;p}\,\omega_{;q}\,(3\,\omega_{;r}\,\omega^{;r}\,R_g{}^{p}{}_{h}{}^{q} + 4\,\omega_{;rg}\,R_h{}^{rpq})) \\
& -\frac{g_{ab}\,g_{cd}\,g_{ef}\,g_{gh}}{241920}\omega_{;p}\,\omega_{;q}\,(6\,\omega_{;r}\,\omega^{;p}\,\omega^{;q}\,\omega^{;r} - 67\,\omega_{;r}\,\omega^{;r}\,\omega^{;pq} \\
&\quad +31\,\omega_{;r}\,\omega^{;q}\,\omega^{;pr} + 102\,\omega_{;r}{}^{q}\,\omega^{;pr} - 6\,\omega_{;r}\,\omega^{;pqr} + 27\,\omega_{;r}\,\omega^{;prq} \\
&\quad +15\,\omega_{;r}\,\omega^{;qrp})
\end{aligned} \tag{A8g}$$

where $\doteq$ denotes equality upon symmetrization.

By using the expansion

$$
\begin{aligned}
e^{\omega+\omega'} &= e^{2\omega}\Big(1-\sqrt{2}\,\sqrt{\sigma}\,p^p\omega_{;p}+\sigma\,p^p\,p^q\,(\omega_{;p}\,\omega_{;q}+\omega_{;pq})\\
&\quad -\frac{\sqrt{2}}{3}\sigma^{\frac{3}{2}}\,p^p\,p^q\,p^r\,(\omega_{;p}\,\omega_{;q}\,\omega_{;r}+3\,\omega_{;p}\,\omega_{;qr}+\omega_{;pqr})\\
&\quad +\frac{1}{6}\sigma^2\,p^p\,p^q\,p^r\,p^s\,(\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;s}+6\,\omega_{;p}\,\omega_{;q}\,\omega_{;rs}+3\,\omega_{;pq}\,\omega_{;rs}\\
&\qquad +4\,\omega_{;p}\,\omega_{;qrs}+\omega_{;pqrs})\\
&\quad -\frac{1}{15\,\sqrt{2}}\sigma^{\frac{5}{2}}\,p^p\,p^q\,p^r\,p^s\,p^t\,(\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;s}\,\omega_{;t}+10\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;st}\\
&\qquad +15\,\omega_{;p}\,\omega_{;qr}\,\omega_{;st}+10\,\omega_{;p}\,\omega_{;q}\,\omega_{;rst}+10\,\omega_{;pq}\,\omega_{;rst}\\
&\qquad +5\,\omega_{;p}\,\omega_{;qrst}+\omega_{;pqrst})\\
&\quad +\frac{1}{90}\sigma^3\,p^p\,p^q\,p^r\,p^s\,p^t\,p^u\,(\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;s}\,\omega_{;t}\,\omega_{;u}\\
&\qquad +15\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;s}\,\omega_{;tu}+45\,\omega_{;p}\,\omega_{;q}\,\omega_{;rs}\,\omega_{;tu}+15\,\omega_{;pq}\,\omega_{;rs}\,\omega_{;tu}\\
&\qquad +20\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega_{;stu}+60\,\omega_{;p}\,\omega_{;qr}\,\omega_{;stu}+10\,\omega_{;pqr}\,\omega_{;stu}\\
&\qquad +15\,\omega_{;p}\,\omega_{;q}\,\omega_{;rstu}+15\,\omega_{;pq}\,\omega_{;rstu}+6\,\omega_{;p}\,\omega_{;qrstu}+\omega_{;pqrstu})\Big)+O\left(\sigma^{\frac{7}{2}}\right)
\end{aligned}
\tag{A9}
$$

we can get the expansion of Σ. Multiplying together the above series and (A6) and subtracting σ, we determine the expansion

$$
\begin{aligned}
\Sigma &= \Sigma^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s+\Sigma^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t+\Sigma^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\\
&\quad +\Sigma^{(7)}_{pqrstuv}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v+\Sigma^{(8)}_{pqrstuvw}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v\sigma^w
\end{aligned}
\tag{A10}
$$

where the expansion tensors are

$$
\Sigma^{(4)}_{abcd} \doteq \frac{g_{ab}}{3}\,(\omega_{;c}\,\omega_{;d}+\omega_{;cd})-\frac{g_{ab}\,g_{cd}}{6}\omega_{;p}\,\omega^{;p}
\tag{A11a}
$$

$$
\Sigma^{(5)}_{abcde} \doteq -\frac{g_{ab}}{3\,\sqrt{2}}\,(2\,\omega_{;c}\,\omega_{;de}+\omega_{;cde})+\frac{g_{ab}\,g_{cd}}{3\,\sqrt{2}}\omega_{;p}\,\omega_{;e}{}^{p}
\tag{A11b}
$$

$$
\begin{aligned}
\Sigma^{(6)}_{abcdef} &\doteq \frac{g_{ab}}{30}\,(\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}+2\,\omega_{;c}\,\omega_{;d}\,\omega_{;ef}+7\,\omega_{;cd}\,\omega_{;ef}\\
&\qquad +6\,\omega_{;c}\,\omega_{;def}+3\,\omega_{;cdef})\\
&\quad -\frac{g_{ab}\,g_{cd}}{90}\,(3\,\omega_{;r}\,\omega_{;e}\,\omega_{;f}\,\omega^{;r}+7\,\omega_{;p}\,\omega^{;p}\,\omega_{;ef}\\
&\qquad +9\,\omega_{;p}\,\omega_{;ef}{}^{p}+12\,\omega_{;p}\,\omega_{;q}\,R_{e}{}^{q}{}_{f}{}^{p})\\
&\quad +\frac{1}{720}8\,g_{ab}\,g_{cd}\,g_{ef}\;\omega_{;p}\,\omega_{;q}\;(\omega^{;p}\,\omega^{;q}-3\,\omega^{;pq})
\end{aligned}
\tag{A11c}
$$

$$
\begin{aligned}
\Sigma^{(7)}_{abcdefg} &\doteq -\frac{g_{ab}}{45\,\sqrt{2}}\,(6\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;fg}+6\,\omega_{;c}\,\omega_{;de}\,\omega_{;fg}+3\,\omega_{;c}\,\omega_{;d}\,\omega_{;efg}\\
&\qquad +15\,\omega_{;cd}\,\omega_{;efg}+4\,\omega_{;c}\,\omega_{;defg}+2\,\omega_{;cdefg})\\
&\quad \frac{g_{ab}\,g_{cd}}{270\,\sqrt{2}}\,(18\,\omega_{;p}\,\omega_{;e}\,\omega^{;p}\,\omega_{;fg}-50\,\omega_{;p}\,\omega_{;e}{}^{p}\,\omega_{;fg}+18\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}{}^{p}\\
&\qquad -24\,\omega_{;e}\,\omega_{;pf}\,\omega_{;g}{}^{p}+68\,\omega_{;p}\,\omega_{;ef}\,\omega_{;g}{}^{p}+21\,\omega_{;p}\,\omega^{;p}\,\omega_{;efg}\\
&\qquad -24\,\omega_{;p}\,\omega_{;e}\,\omega_{;fg}{}^{p}+24\,\omega_{;pe}\,\omega_{;fg}{}^{p}+6\,\omega_{;pe}\,\omega_{;f}{}^{p}{}_{g}\\
&\qquad +12\,\omega_{;p}\,\omega_{;efg}{}^{p}+6\,\omega_{;p}\,\omega_{;ef}{}^{p}{}_{g}-6\,\omega_{;p}\,\omega_{;e}{}^{p}{}_{fg}\\
&\qquad +27\,\omega_{;p}\,\omega_{;q}\,R_{e}{}^{p}{}_{f}{}^{q}{}_{;g}-12\,\omega_{;p}\;(2\,\omega_{;q}\,\omega_{;e}-7\,\omega_{;qe})\;R_{f}{}^{p}{}_{g}{}^{q})\\
&\quad -\frac{g_{ab}\,g_{cd}\,g_{ef}}{180\,\sqrt{2}}\;\omega_{;p}\,(8\,\omega_{;q}\,\omega^{;p}\,\omega_{;g}{}^{q}-3\,\omega_{;q}{}^{p}\,\omega_{;g}{}^{q}-9\,\omega_{;qg}\,\omega^{;pq}\\
&\qquad -4\,\omega_{;q}\,\omega_{;g}{}^{pq}-2\,\omega_{;q}\,\omega^{;pq}{}_{g}+2\,\omega_{;q}\,\omega_{;r}\,R_{g}{}^{pqr})
\end{aligned}
\tag{A11d}
$$

$$
\begin{aligned}
\Sigma^{(8)}_{abcdefgh} &\doteq \frac{g_{ab}}{630}\,\{\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega_{;h}+3\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}\,\omega_{;gh}\\
&\qquad +69\,\omega_{;c}\,\omega_{;d}\,\omega_{;ef}\,\omega_{;gh}+27\,\omega_{;cd}\,\omega_{;ef}\,\omega_{;gh}\\
&\qquad +26\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;fgh}+66\,\omega_{;c}\,\omega_{;de}\,\omega_{;fgh}\\
&\qquad +40\,\omega_{;cde}\,\omega_{;fgh}+13\,\omega_{;c}\,\omega_{;d}\,\omega_{;efgh}+53\,\omega_{;cd}\,\omega_{;efgh}\\
&\qquad +10\,\omega_{;c}\,\omega_{;defgh}+5\,\omega_{;cdefgh}\}\\
&\quad -\frac{g_{ab}\,g_{cd}}{7560}\,(12\;(8\,\omega_{;pe}\,\omega_{;fg}{}^{p}{}_{h}-4\,\omega_{;pe}\,\omega_{;f}{}^{p}{}_{gh}+5\,\omega_{;p}\,\omega_{;efgh}{}^{p})
\end{aligned}
$$

$$
\begin{aligned}
&+12\,\omega_{;p}\,\left(5\,\omega_{;efgh}{}^{p}+3\,\omega_{;efg}{}^{p}{}_{h}+\omega_{;ef}{}^{p}{}_{gh}-4\,\omega_{;e}{}^{p}{}_{fgh}\right)\\
&-84\,\omega_{;pef}\,\left(\omega_{;gh}{}^{p}-\omega_{;g}{}^{p}{}_{h}\right)\\
&+9\,\left(15\,\omega_{;efp}\,\omega_{;gh}{}^{p}+16\,\omega_{;pe}\,\omega_{;fgh}{}^{p}\right)\\
&+6\,\omega_{;p}\,(66\,\omega_{;e}{}^{p}\,\omega_{;fgh}-12\,\omega_{;ef}\,\omega_{;gh}{}^{p}+12\,\omega_{;ef}\,\omega_{;g}{}^{p}{}_{h}\\
&\quad+29\,\omega^{;p}\,\omega_{;efgh}-28\,\omega_{;e}\,\omega_{;fgh}{}^{p}-20\,\omega_{;e}\,\omega_{;fg}{}^{p}{}_{h}\\
&\quad+16\,\omega_{;e}\,\omega_{;f}{}^{p}{}_{gh})-24\,\omega_{;e}\,\omega_{;pf}\,\left(19\,\omega_{;gh}{}^{p}+4\,\omega_{;g}{}^{p}{}_{h}\right)\\
&-8\,\omega_{;pe}\,\left(25\,\omega_{;f}{}^{p}\,\omega_{;gh}-21\,\omega_{;fg}\,\omega_{;h}{}^{p}\right)\\
&+4\,\omega_{;p}\,\omega_{;e}\,(134\,\omega_{;f}{}^{p}\,\omega_{;gh}-60\,\omega_{;fg}\,\omega_{;h}{}^{p}+33\,\omega^{;p}\,\omega_{;fgh}\\
&\quad+39\,\omega_{;f}\,\omega_{;gh}{}^{p}+12\,\omega_{;f}\,\omega_{;g}{}^{p}{}_{h})+266\,\omega_{;p}\,\omega^{;p}\,\omega_{;ef}\,\omega_{;gh}\\
&+208\,\omega_{;e}\,\omega_{;f}\,\omega_{;pg}\,\omega_{;h}{}^{p}+18\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\omega_{;g}\,\omega_{;h}\,\omega^{;p}\\
&+4\,\omega_{;p}\,\omega_{;e}\,\omega_{;f}\,\left(17\,\omega^{;p}\,\omega_{;gh}-16\,\omega_{;g}\,\omega_{;h}{}^{p}\right)\\
&+16\,(11\,\omega_{;p}\,\omega_{;q}\,\omega_{;e}\,\omega_{;f}-74\,\omega_{;p}\,\omega_{;e}\,\omega_{;qf}+41\,\omega_{;pe}\,\omega_{;qf}\\
&\quad+3\,\omega_{;p}\,\omega_{;q}\,\omega_{;ef}+21\,\omega_{;p}\,\omega_{;qef}+30\,\omega_{;p}\,\omega_{;efq})\,R_{g}{}^{p}{}_{h}{}^{q}\\
&-312\,\omega_{;p}\,\omega_{;q}\,\omega_{;e}\,R_{f}{}^{p}{}_{g}{}^{q}{}_{;h}-912\,\omega_{;p}\,R_{qef}{}^{p}{}_{;g}\,\omega_{;h}{}^{q}\\
&+180\,\omega_{;p}\,\omega_{;q}\,R_{e}{}^{p}{}_{f}{}^{q}{}_{;gh}-336\,\omega_{;p}\,\omega_{;q}\,R_{ref}{}^{p}\,R_{g}{}^{r}{}_{h}{}^{q}\}\\
\frac{g_{ab}\,g_{cd}\,g_{ef}}{15120}&\{-256\,\omega_{;pq}\,\omega_{;g}{}^{p}\,\omega_{;h}{}^{q}-300\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;gh}{}^{q}\\
&-24\,\omega_{;p}\,\omega_{;qg}\,\left(4\,\omega_{;h}{}^{pq}+11\,\omega_{;h}{}^{qp}+7\,\omega^{;pq}{}_{h}\right)\\
&-24\,\omega_{;p}\,\omega_{;q}\,\left(6\,\omega_{;gh}{}^{pq}-3\,\omega_{;g}{}^{p}{}_{h}{}^{q}+\omega_{;g}{}^{pq}{}_{h}+2\,\omega^{;pq}{}_{gh}\right)\\
&+8\,\omega_{;p}\,\omega^{;p}\,\left(20\,\omega_{;qg}\,\omega_{;h}{}^{q}+3\,\omega_{;q}\,\omega_{;gh}{}^{q}+18\,\omega_{;q}\,\omega_{;g}{}^{q}{}_{h}\right)\\
&+24\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\left(2\,\omega_{;h}{}^{pq}+3\,\omega^{;pq}{}_{h}\right)+256\,\omega_{;p}\,\omega_{;g}\,\omega_{;q}{}^{p}\,\omega_{;h}{}^{q}\\
&+16\,\omega_{;p}\,\omega_{;q}\,\left(29\,\omega_{;g}{}^{p}\,\omega_{;h}{}^{q}-18\,\omega_{;gh}\,\omega^{;pq}\right)\\
&+8\,\omega_{;p}\,\omega_{;q}\,\left(9\,\omega^{;p}\,\omega^{;q}\,\omega_{;gh}-12\,\omega_{;g}\,\omega^{;p}\,\omega_{;h}{}^{q}-5\,\omega_{;g}\,\omega_{;h}\,\omega^{;pq}\right)\\
&+24\,\omega_{;p}\,\omega_{;q}\,\omega_{;g}\,\omega_{;h}\,\omega^{;p}\,\omega^{;q}\\
&-3\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\left(15\,R_{g}{}^{p}{}_{h}{}^{q;r}+45\,R_{g}{}^{p}{}_{h}{}^{r;q}-26\,R_{g}{}^{pqr}{}_{;h}\right)\\
&+8\,\omega_{;p}\,\omega_{;q}\,\left(9\,\omega_{;r}\,\omega_{;g}\,R_{h}{}^{pqr}+18\,\omega_{;rg}\,R_{h}{}^{pqr}+16\,\omega_{;rg}\,R_{h}{}^{rpq}\right)\\
&+48\,\omega_{;p}\,\omega_{;q}\,\left(2\,\omega_{;r}\,\omega^{;p}-15\,\omega_{;r}{}^{p}\right)\,R_{g}{}^{q}{}_{h}{}^{r}\}\\
-\frac{g_{ab}\,g_{cd}\,g_{ef}\,g_{gh}}{2520}&\,\omega_{;p}\,\omega_{;q}\,(\omega_{;r}\,\omega^{;p}\,\omega^{;q}\,\omega^{;r}-6\,\omega_{;r}\,\omega^{;p}\,\omega^{;qr}\\
&\qquad+17\,\omega_{;r}{}^{p}\,\omega^{;qr}+6\,\omega_{;r}\,\omega^{;pqr})
\end{aligned}
\tag{A11e}
$$

Turning to the last expansion we need, we recall the Van Vleck-Morette determinant on the optical metric, $\bar{\Delta}^{1/2}$, satisfies the differential equation

$$\bar{\Delta}^{1/2}\left(4-\bar{\Box}\bar{\sigma}\right)-2\bar{\Delta}^{1/2}{}_{|p}\bar{\sigma}^{|p}=0 \tag{A12}$$

Using the conformal transformation property

$$\bar{\Box}\bar{\sigma}=e^{2\omega}\left(\Box\bar{\sigma}-2\omega^{;p}\bar{\sigma}_{;p}\right) \tag{A13}$$

we determine $\bar{\Delta}^{1/2}$ satisfies the equation

$$\bar{\Delta}^{1/2}\left(4-e^{2\omega}\left(\Box\bar{\sigma}-2\omega^{;p}\bar{\sigma}_{;p}\right)\right)-2e^{2\omega}\bar{\Delta}^{1/2}{}_{;p}\bar{\sigma}^{;p}=0 \tag{A14}$$

on the physical metric. Since we have the expansion for $\bar{\sigma}$, we need only assume the expansion

$$
\begin{aligned}
\bar{\Delta}^{1/2} \;=\; & 1+\bar{\Delta}^{(2)}_{pq}\sigma^{p}\sigma^{q}+\bar{\Delta}^{(3)}_{pqr}\sigma^{p}\sigma^{q}\sigma^{r}+\bar{\Delta}^{(4)}_{pqrs}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}+\bar{\Delta}^{(5)}_{pqrst}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}\sigma^{t}\\
&+\bar{\Delta}^{(6)}_{pqrstu}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}\sigma^{t}\sigma^{u}
\end{aligned}
\tag{A15}
$$

and substitute this and (A6) into (A14) and solve for the expansion tensors. Following the now well defined method outlined above, they are found to be

$$\bar{\Delta}^{(2)}_{ab} \;\doteq\; \Delta^{(2)}_{ab}+\frac{1}{6}\left(\omega_{;a}\,\omega_{;b}+\omega_{;ab}\right)-\frac{g_{ab}}{12}\left(2\,\omega_{;p}\,\omega^{;p}-\omega_{;p}{}^{p}\right) \tag{A16a}$$

$$\bar{\Delta}^{(3)}_{abc} \;\doteq\; \Delta^{(3)}_{abc}-\frac{1}{12}\left(2\,\omega_{;a}\,\omega_{;bc}+\omega_{;abc}\right)+\frac{g_{ab}}{24}\left(4\,\omega_{;p}\,\omega_{;c}{}^{p}-\omega_{;p}{}^{p}{}_{c}\right) \tag{A16b}$$

$$
\begin{aligned}
\bar{\Delta}^{(4)}_{abcd} \;\doteq\; & \Delta^{(4)}_{abcd}+\frac{1}{72}\left(\omega_{;c}\,\omega_{;d}+\omega_{;cd}\right)\,R_{ab}+\frac{1}{120}\left(7\,\omega_{;ab}\,\omega_{;cd}+6\,\omega_{;a}\,\omega_{;bcd}\right)\\
&+\frac{1}{120}\left(\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}+2\,\omega_{;a}\,\omega_{;b}\,\omega_{;cd}+3\,\omega_{;abcd}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g_{ab}}{720}\{6\,\omega_{;c}\,(\omega_{;d}\,\omega_{;p}{}^{p}-\omega_{;p}{}^{p}{}_{d})\\
&\qquad+3\,(4\,\omega_{;p}{}^{p}\,\omega_{;cd}-12\,\omega_{;pc}\,\omega_{;d}{}^{p}+3\,\omega_{;p}{}^{p}{}_{cd})\\
&\qquad-12\,\omega_{;p}\,(\omega_{;c}\,\omega_{;d}\,\omega^{;p}+2\,\omega^{;p}\,\omega_{;cd}-2\,\omega_{;c}\,\omega_{;d}{}^{p}-\omega_{;cd}{}^{p}+4\,\omega_{;c}{}^{p}{}_{d})\\
&\qquad+6\,\omega_{;p}\,(R_{cd}{}^{;p}-R_{c}{}^{p}{}_{;d})-5\,(2\,\omega_{;p}\,\omega^{;p}-\omega_{;p}{}^{p})\,R_{cd}\\
&\qquad-2\,(2\,\omega_{;p}\,\omega_{;d}-\omega_{;pd})\,R_{c}{}^{p}+4\,(\omega_{;p}\,\omega_{;q}+\omega_{;pq})\,R_{c}{}^{p}{}_{d}{}^{q}\}\\
&+\frac{g_{ab}\,g_{cd}}{1440}\{2\,\omega_{;p}\,\omega_{;q}\,(6\,\omega^{;p}\,\omega^{;q}-14\,\omega^{;pq}+3\,R^{pq})\\
&\qquad-14\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}+5\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}+4\,\omega_{;pq}\,\omega^{;pq}+12\,\omega_{;p}\,\omega_{;q}{}^{qp}\}
\end{aligned}
\tag{A16c}
$$

$$
\begin{aligned}
\bar\Delta^{(5)}_{abcde} \doteq\; &\Delta^{(5)}_{abcde}-\frac{1}{360}\,(6\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;de}+6\,\omega_{;a}\,\omega_{;bc}\,\omega_{;de}+3\,\omega_{;a}\,\omega_{;b}\,\omega_{;cde}\\
&\quad+15\,\omega_{;ab}\,\omega_{;cde}+4\,\omega_{;a}\,\omega_{;bcde}+2\,\omega_{;abcde})\\
&-\frac{1}{144}\,(R_{cd;e}\,(\omega_{;a}\,\omega_{;b}+\omega_{;ab})+(2\,\omega_{;a}\,\omega_{;bc}+\omega_{;abc})\,R_{de})\\
&-\frac{g_{ab}}{1440}\{6\,\omega_{;c}\,(2\,\omega_{;p}{}^{p}\,\omega_{;de}+\omega_{;d}\,\omega_{;p}{}^{p}{}_{e})\\
&\qquad+2\,(3\,\omega_{;cd}\,\omega_{;p}{}^{p}{}_{e}+6\,\omega_{;p}{}^{p}\,\omega_{;cde}-3\,\omega_{;c}\,\omega_{;p}{}^{p}{}_{de}+2\,\omega_{;p}{}^{p}{}_{cde})\\
&\qquad+24\,(\omega_{;c}\,\omega_{;pd}\,\omega_{;e}{}^{p}+3\,\omega_{;pc}\,\omega_{;de}{}^{p}-5\,\omega_{;pc}\,\omega_{;d}{}^{p}{}_{e})\\
&\qquad+2\,\omega_{;p}\,(5\,\omega_{;cde}{}^{p}+\omega_{;cd}{}^{p}{}_{e}-14\,\omega_{;c}{}^{p}{}_{de})\\
&\qquad-24\,\omega_{;p}\,\omega_{;c}\,(\omega^{;p}\,\omega_{;de}+\omega_{;d}\,\omega_{;e}{}^{p})\\
&\qquad-8\,\omega_{;p}\,(3\,\omega_{;cd}\,\omega_{;e}{}^{p}+3\,\omega^{;p}\,\omega_{;cde}-10\,\omega_{;c}\,\omega_{;de}{}^{p}+7\,\omega_{;c}\,\omega_{;d}{}^{p}{}_{e})\\
&\qquad+4\,(20\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}+22\,\omega_{;q}\,\omega_{;pc}+\omega_{;pqc}+10\,\omega_{;p}\,R_{qc})\,R_{d}{}^{p}{}_{e}{}^{q}\\
&\qquad+4\,R_{c}{}^{p}{}_{d}{}^{q}{}_{;e}\,(\omega_{;p}\,\omega_{;q}+\omega_{;pq})+\omega_{;p}\,(5\,R_{cd;e}{}^{p}+R_{cd}{}^{;p}{}_{e}-6\,R_{c}{}^{p}{}_{;de})\\
&\qquad-2\,(2\,\omega_{;c}\,\omega_{;pd}+2\,\omega_{;p}\,\omega_{;cd}+14\,\omega_{;pcd}-15\,\omega_{;cdp})\,R_{e}{}^{p}\\
&\qquad-5\,R_{cd;e}\,(2\,\omega_{;p}\,\omega^{;p}-\omega_{;p}{}^{p})-5\,(4\,\omega_{;p}\,\omega_{;c}{}^{p}-\omega_{;p}{}^{p}{}_{c})\,R_{de}\\
&\qquad-4\,R_{d}{}^{p}{}_{;e}\,(\omega_{;p}\,\omega_{;c}+\omega_{;pc})+6\,R_{de}{}^{;p}\,\omega_{;pc}\}\\
&-\frac{g_{ab}\,g_{cd}}{1440}\{2\,\omega_{;p}\,\omega_{;q}\,(12\,\omega^{;p}\,\omega_{;e}{}^{q}-10\,\omega_{;e}{}^{pq}+3\,\omega^{;pq}{}_{e})\\
&\qquad-14\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;e}{}^{p}-28\,\omega_{;p}\,\omega_{;qe}\,\omega^{;pq}-7\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}{}_{e}\\
&\qquad5\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}{}_{e}+6\,\omega_{;pe}\,\omega_{;q}{}^{qp}+4\,\omega_{;pq}\,\omega^{;pq}{}_{e}+5\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{e}{}^{p}\\
&\qquad+\omega_{;p}\,\omega_{;q}{}^{qp}{}_{e}+3\,\omega_{;p}\,(\omega_{;q}\,R^{pq}{}_{;e}+2\,\omega_{;qe}\,R^{pq})\}
\end{aligned}
\tag{A16d}
$$

$$
\bar\Delta^{(6)}_{abcdef} \doteq \Delta^{(6)}_{abcdef}+\bar\Delta^{(6')}_{abcdef}+g_{ab}\bar\Delta^{(6')}_{cdef}+g_{ab}g_{cd}\bar\Delta^{(6')}_{ef}+g_{ab}g_{cd}g_{ef}\bar\Delta^{(6')}
\tag{A16e}
$$

$$
\begin{aligned}
181440\bar\Delta^{(6')}_{abcdef} =\; &-22556\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega_{;e}\,\omega_{;f}-67668\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega_{;ef}\\
&-48444\,\omega_{;a}\,\omega_{;b}\,\omega_{;cd}\,\omega_{;ef}-10296\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;def}\\
&-2612\,\omega_{;ab}\,\omega_{;cd}\,\omega_{;ef}-11016\,\omega_{;a}\,\omega_{;bc}\,\omega_{;def}-108\,\omega_{;a}\,\omega_{;b}\,\omega_{;cdef}\\
&+900\,\omega_{;abc}\,\omega_{;def}+1332\,\omega_{;ab}\,\omega_{;cdef}+360\,\omega_{;a}\,\omega_{;bcdef}\\
&+180\,\omega_{;abcdef}\\
&+90\,(\omega_{;a}\,\omega_{;b}+\omega_{;ab})\,R_{cd;ef}-5040\,\omega_{;a}\,R_{d}{}^{p}{}_{e}{}^{q}{}_{;f}\,R_{pbqc}\\
&-90\,R_{de;f}\,(56\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}+54\,\omega_{;a}\,\omega_{;bc}-\omega_{;abc})\\
&-30\,(751\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}+1166\,\omega_{;a}\,\omega_{;b}\,\omega_{;cd}+73\,\omega_{;ab}\,\omega_{;cd}\\
&\quad+162\,\omega_{;a}\,\omega_{;bcd}-3\,\omega_{;abcd})\,R_{ef}\\
&+105\,(\omega_{;a}\,\omega_{;b}+\omega_{;ab})\,R_{cd}\,R_{ef}\\
&-1260\,(9\,\omega_{;a}\,\omega_{;b}+\omega_{;ab})\,R_{pcqd}\,R_{e}{}^{p}{}_{f}{}^{q}
\end{aligned}
\tag{A17a}
$$

$$
\begin{aligned}
120960\bar\Delta^{(6')}_{abcd} =\; &31672\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega^{;p}-9116\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega_{;p}{}^{p}\\
&+41376\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}\,\omega_{;cd}+43936\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}{}^{p}\\
&-15568\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^{p}\,\omega_{;cd}+4088\,\omega_{;p}\,\omega^{;p}\,\omega_{;ab}\,\omega_{;cd}\\
&+15152\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc}\,\omega_{;d}{}^{p}+32416\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}\,\omega_{;d}{}^{p}\\
&-2664\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;p}{}^{p}{}_{d}+4320\,\omega_{;p}\,\omega_{;a}\,\omega^{;p}\,\omega_{;bcd}\\
&+2256\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;cd}{}^{p}+4896\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}{}^{p}{}_{d}\\
&-1196\,\omega_{;p}{}^{p}\,\omega_{;ab}\,\omega_{;cd}+3056\,\omega_{;pa}\,\omega_{;bc}\,\omega_{;d}{}^{p}-2952\,\omega_{;a}\,\omega_{;bc}\,\omega_{;p}{}^{p}{}_{d}\\
&-2736\,\omega_{;a}\,\omega_{;p}{}^{p}\,\omega_{;bcd}+1152\,\omega_{;p}\,\omega_{;a}{}^{p}\,\omega_{;bcd}+3264\,\omega_{;a}\,\omega_{;pb}\,\omega_{;cd}{}^{p}\\
&+8256\,\omega_{;p}\,\omega_{;ab}\,\omega_{;cd}{}^{p}+4800\,\omega_{;a}\,\omega_{;pb}\,\omega_{;c}{}^{p}{}_{d}-5424\,\omega_{;p}\,\omega_{;ab}\,\omega_{;c}{}^{p}{}_{d}\\
&-12\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^{p}{}_{cd}-408\,\omega_{;p}\,\omega^{;p}\,\omega_{;abcd}+1848\,\omega_{;p}\,\omega_{;a}\,\omega_{;bcd}{}^{p}\\
&-264\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}{}^{p}{}_{d}-624\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^{p}{}_{cd}+1560\,\omega_{;pab}\,\omega_{;cd}{}^{p}
\end{aligned}
$$

$$
\begin{aligned}
&+1260\,\omega_{;abp}\,\omega_{;cd}{}^{p} - 3180\,\omega_{;pab}\,\omega_{;c}{}^{p}{}_{d} - 156\,\omega_{;ab}\,\omega_{;p}{}^{p}{}_{cd} \\
&+108\,\omega_{;p}{}^{p}\,\omega_{;abcd} + 1728\,\omega_{;pa}\,\omega_{;bcd}{}^{p} - 192\,\omega_{;pa}\,\omega_{;bc}{}^{p}{}_{d} \\
&-2112\,\omega_{;pa}\,\omega_{;b}{}^{p}{}_{cd} - 144\,\omega_{;a}\,\omega_{;p}{}^{p}{}_{bcd} + 216\,\omega_{;p}\,\omega_{;abcd}{}^{p} \\
&+96\,\omega_{;p}\,\omega_{;abc}{}^{p}{}_{d} - 24\,\omega_{;p}\,\omega_{;ab}{}^{p}{}_{cd} - 528\,\omega_{;p}\,\omega_{;a}{}^{p}{}_{bcd} \\
&+60\,\omega_{;p}{}^{p}{}_{abcd} \\
&+12\,\omega_{;p}\,\left(9\,R_{ab;cd}{}^{p} + 4\,R_{ab;c}{}^{p}{}_{d} - R_{ab}{}^{;p}{}_{cd} - 12\,R_{a}{}^{p}{}_{;bcd}\right) \\
&-6\,\left(10\,\omega_{;p}\,\omega^{;p} - 21\,\omega_{;p}{}^{p}\right)\,R_{ab;cd} + 12\,\left(3\,\omega_{;p}\,\omega_{;a} + 16\,\omega_{;pa}\right)\,R_{bc;d}{}^{p} \\
&-12\,\left(5\,\omega_{;p}\,\omega_{;a} - 6\,\omega_{;pa}\right)\,R_{bc}{}^{;p}{}_{d} - 72\,\left(\omega_{;p}\,\omega_{;a} + 3\,\omega_{;pa}\right)\,R_{b}{}^{p}{}_{;cd} \\
&+3360\,\omega_{;p}\,\omega_{;a}\,\omega^{;p}\,R_{bc;d} + 288\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,R_{cd}{}^{;p} \\
&+696\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,R_{c}{}^{p}{}_{;d} - 96\,\omega_{;a}\,R_{cd}{}^{;p}\,\omega_{;pb} - 96\,\omega_{;a}\,R_{c}{}^{p}{}_{;d}\,\omega_{;pb} \\
&+396\,\omega_{;p}\,R_{cd}{}^{;p}\,\omega_{;ab} + 384\,\omega_{;p}\,R_{c}{}^{p}{}_{;d}\,\omega_{;ab} + 600\,\omega_{;p}\,R_{bc;d}\,\omega_{;a}{}^{p} \\
&-480\,R_{cd}{}^{;p}\,\omega_{;pab} - 1320\,R_{c}{}^{p}{}_{;d}\,\omega_{;pab} + 210\,R_{bc;d}\,\omega_{;p}{}^{p}{}_{a} \\
&+630\,R_{cd}{}^{;p}\,\omega_{;abp} + 1260\,R_{c}{}^{p}{}_{;d}\,\omega_{;abp} + 84\,\omega_{;p}\,\left(R_{cd}{}^{;p} - R_{c}{}^{p}{}_{;d}\right)\,R_{ab} \\
&-840\,\omega_{;p}\,\left(R_{cd}{}^{;q} + 2\,R_{c}{}^{q}{}_{;d}\right)\,R_{qab}{}^{p} - 96\,\left(\omega_{;p}\,\omega_{;q} + \omega_{;pq}\right)\,R_{a}{}^{p}{}_{b}{}^{q}{}_{;cd} \\
&-12\,R_{b}{}^{p}{}_{c}{}^{q}{}_{;d}\,(80\,\omega_{;p}\,\omega_{;q}\,\omega_{;a} + 222\,\omega_{;a}\,\omega_{;pq} - 112\,\omega_{;q}\,\omega_{;pa} \\
&\quad +15\,\omega_{;pqa} - 70\,\omega_{;q}\,R_{pa}) + 348\,\omega_{;p}\,\left(R_{b}{}^{q}{}_{c}{}^{r}{}_{;d} + R_{b}{}^{r}{}_{c}{}^{q}{}_{;d}\right)\,R_{qar}{}^{p} \\
&+24\,\omega_{;p}\,\left(11\,R_{c}{}^{q}{}_{d}{}^{r;p} + 15\,R_{c}{}^{qpr}{}_{;d} + 15\,R_{c}{}^{rpq}{}_{;d}\right)\,R_{qarb} \\
&+2\,(8300\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p} + 42\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^{p} + 1688\,\omega_{;p}\,\omega^{;p}\,\omega_{;ab} \\
&\quad +84\,\omega_{;p}{}^{p}\,\omega_{;ab} + 6504\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^{p} + 132\,\omega_{;pa}\,\omega_{;b}{}^{p} \\
&\quad -42\,\omega_{;a}\,\omega_{;p}{}^{p}{}_{b} + 1188\,\omega_{;p}\,\omega_{;ab}{}^{p} - 768\,\omega_{;p}\,\omega_{;a}{}^{p}{}_{b} + 63\,\omega_{;p}{}^{p}{}_{ab})\,R_{cd} \\
&+24\,(246\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c} + 19\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc} + 304\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc} \\
&\quad +11\,\omega_{;pa}\,\omega_{;bc} + 44\,\omega_{;a}\,\omega_{;pbc} + 26\,\omega_{;p}\,\omega_{;abc} \\
&\quad -49\,\omega_{;a}\,\omega_{;bcp} - 19\,\omega_{;pabc} - 7\,\omega_{;abpc} + 28\,\omega_{;abcp})\,R_{d}{}^{p} \\
&+56\,\left(37\,\omega_{;p}\,\omega_{;q} + \omega_{;pq}\right)\,R_{ab}\,R_{c}{}^{p}{}_{d}{}^{q} \\
&-1120\,\left(\omega_{;p}\,\omega_{;a} - 2\,\omega_{;pa}\right)\,R_{qb}\,R_{c}{}^{q}{}_{d}{}^{p} \\
&-4\,\{1844\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega_{;b} + 2308\,\omega_{;a}\,\omega_{;b}\,\omega_{;pq} + 256\,\omega_{;q}\,\omega_{;a}\,\omega_{;pb} \\
&\quad -1144\,\omega_{;pb}\,\omega_{;qa} - 1208\,\omega_{;p}\,\omega_{;a}\,\omega_{;qb} + 56\,\omega_{;pa}\,\omega_{;qb} \\
&\quad -1520\,\omega_{;p}\,\omega_{;q}\,\omega_{;ab} + 316\,\omega_{;pq}\,\omega_{;ab} + 660\,\omega_{;a}\,\omega_{;pqb} \\
&\quad -768\,\omega_{;q}\,\omega_{;pab} - 18\,\omega_{;p}\,\omega_{;qab} - 420\,\omega_{;p}\,\omega_{;abq} + 24\,\omega_{;pqab} \\
&\quad -7\,\left(30\,\omega_{;r}\,\omega^{;r} + \omega_{;r}{}^{r}\right)\,R_{paqb} \\
&\quad -12\,\left(59\,\omega_{;r}\,\omega_{;a} + 2\,\omega_{;ra}\right)\,R_{pbq}{}^{r} \\
&\quad -4\,\left(191\,\omega_{;r}\,\omega_{;a} + 20\,\omega_{;ra}\right)\,R_{p}{}^{r}{}_{qb} \\
&\quad -\left(\left(19\,\omega_{;p}\,\omega_{;r} + 16\,\omega_{;pr}\right)\,R_{qab}{}^{r}\right)\}\,R_{c}{}^{p}{}_{d}{}^{q}
\end{aligned}
\tag{A17b}
$$

$$
\begin{aligned}
120960\bar{\Delta}^{(6')}_{ab} \;=\; &-18232\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}\,\omega^{;q} + 10364\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}\,\omega_{;q}{}^{q} \\
&-8\,\omega_{;p}\,\omega_{;q}\,\left(520\,\omega^{;p}\,\omega^{;q}\,\omega_{;ab} + 3518\,\omega_{;a}\,\omega^{;p}\,\omega_{;b}{}^{q} + 1353\,\omega_{;a}\,\omega_{;b}\,\omega^{;pq}\right) \\
&+14\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q} + 2720\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}\,\omega_{;ab} \\
&+9888\,\omega_{;p}\,\omega_{;a}\,\omega_{;q}{}^{q}\,\omega_{;b}{}^{p} - 2632\,\omega_{;p}\,\omega^{;p}\,\omega_{;qa}\,\omega_{;b}{}^{q} \\
&-4576\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}{}^{p}\,\omega_{;b}{}^{q} - 2368\,\omega_{;a}\,\omega_{;b}\,\omega_{;pq}\,\omega^{;pq} \\
&-9152\,\omega_{;p}\,\omega_{;a}\,\omega_{;qb}\,\omega^{;pq} - 3440\,\omega_{;p}\,\omega_{;q}\,\omega_{;ab}\,\omega^{;pq} \\
&+2700\,\omega_{;p}\,\omega_{;a}\,\omega^{;p}\,\omega_{;q}{}^{q}{}_{b} + 240\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;q}{}^{qp} \\
&-3024\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;ab}{}^{q} + 1104\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;a}{}^{q}{}_{b} \\
&-96\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega_{;b}{}^{pq} - 2712\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega^{;pq}{}_{b} \\
&+98\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}\,\omega_{;ab} + 408\,\omega_{;pa}\,\omega_{;q}{}^{q}\,\omega_{;b}{}^{p} - 2096\,\omega_{;pq}\,\omega_{;a}{}^{p}\,\omega_{;b}{}^{q} \\
&+784\,\omega_{;pa}\,\omega_{;qb}\,\omega^{;pq} - 424\,\omega_{;pq}\,\omega_{;ab}\,\omega^{;pq} - 84\,\omega_{;a}\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}{}_{b} \\
&+756\,\omega_{;p}\,\omega_{;a}{}^{p}\,\omega_{;q}{}^{q}{}_{b} - 120\,\omega_{;a}\,\omega_{;pb}\,\omega_{;q}{}^{qp} + 444\,\omega_{;p}\,\omega_{;ab}\,\omega_{;q}{}^{qp} \\
&+1656\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;ab}{}^{p} - 2316\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;ab}{}^{q} - 648\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;a}{}^{p}{}_{b} \\
&+504\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;a}{}^{q}{}_{b} + 864\,\omega_{;p}\,\omega_{;qa}\,\omega_{;b}{}^{pq} - 2928\,\omega_{;p}\,\omega_{;qa}\,\omega_{;b}{}^{qp} \\
&-1944\,\omega_{;a}\,\omega_{;pq}\,\omega^{;pq}{}_{b} - 1104\,\omega_{;p}\,\omega_{;qa}\,\omega^{;pq}{}_{b} + 30\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}{}_{ab} \\
&-72\,\omega_{;p}\,\omega_{;a}\,\omega_{;q}{}^{q}{}_{b}{}^{p} - 24\,\omega_{;p}\,\omega_{;a}\,\omega_{;q}{}^{qp}{}_{b} + 108\,\omega_{;p}\,\omega_{;q}\,\omega_{;ab}{}^{pq} \\
&-648\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}{}^{p}{}_{b}{}^{q} - 120\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}{}^{pq}{}_{b} + 132\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}{}_{ab} \\
&+105\,\omega_{;p}{}^{p}{}_{a}\,\omega_{;q}{}^{q}{}_{b} - 480\,\omega_{;pab}\,\omega_{;q}{}^{qp} + 630\,\omega_{;p}{}^{pq}\,\omega_{;abq} - 90\,\omega_{;pqa}\,\omega^{;pq}{}_{b} \\
&+126\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}{}_{ab} + 192\,\omega_{;pa}\,\omega_{;q}{}^{q}{}_{b}{}^{p} + 72\,\omega_{;pa}\,\omega_{;q}{}^{qp}{}_{b} - 96\,\omega_{;pq}\,\omega^{;pq}{}_{ab} \\
&+108\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{ab}{}^{p} + 48\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{a}{}^{p}{}_{b} - 12\,\omega_{;p}\,\omega_{;q}{}^{qp}{}_{ab} \\
&+18\,\omega_{;p}\,\omega_{;q}\,\left(3\,R_{ab}{}^{;pq} - 6\,R_{a}{}^{p}{}_{;b}{}^{q} + 2\,R_{a}{}^{p;q}{}_{b} + 5\,R^{pq}{}_{;ab}\right) \\
&-12\,\omega_{;p}\,\omega_{;q}\,\left(\omega^{;p}\,\left(26\,R_{ab}{}^{;q} + 53\,R_{a}{}^{q}{}_{;b}\right) + 2\,\omega_{;a}\,\left(2\,R_{b}{}^{p;q} + R^{pq}{}_{;b}\right)\right) \\
&-24\,\omega_{;p}\,\left(R_{b}{}^{p;q} - 2\,R_{b}{}^{q;p} - 11\,R^{pq}{}_{;b}\right)\,\omega_{;qa} \\
&+6\,\omega_{;p}\,\left(\left(17\,R_{ab}{}^{;q} - 18\,R_{a}{}^{q}{}_{;b}\right)\,\omega_{;q}{}^{p} + 14\,\left(R_{ab}{}^{;p} - R_{a}{}^{p}{}_{;b}\right)\,\omega_{;q}{}^{q}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+24\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\left(11\,R_a{}^p{}_b{}^{q;r}-30\,R_a{}^{pqr}{}_{;b}\right)\\
&+24\,\omega_{;p}\,\left(11\,R_a{}^q{}_b{}^{r;p}+30\,R_a{}^{qpr}{}_{;b}\right)\,\omega_{;qr}+42\,\omega_{;p}\,\omega_{;q}\,R_{ab}\,R^{pq}\\
&-(1580\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}+98\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^q-35\,\omega_{;p}{}^p\,\omega_{;q}{}^q\\
&\quad+964\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}-28\,\omega_{;pq}\,\omega^{;pq}-84\,\omega_{;p}\,\omega_{;q}{}^{qp})\,R_{ab}\\
&-4\,(1470\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega^{;q}+101\,\omega_{;q}\,\omega^{;q}\,\omega_{;pa}+16\,\omega_{;q}\,\omega_{;a}\,\omega_{;p}{}^q\\
&\quad+14\,\omega_{;p}\,\omega_{;a}\,\omega_{;q}{}^q-7\,\omega_{;pa}\,\omega_{;q}{}^q+736\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}{}^q-2\,\omega_{;pq}\,\omega_{;a}{}^q\\
&\quad+180\,\omega_{;q}\,\omega_{;pa}{}^q+24\,\omega_{;q}\,\omega_{;p}{}^q{}_a-210\,\omega_{;q}\,\omega_{;a}{}^q{}_p)\,R_b{}^p\\
&-4\,(89\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega_{;b}+28\,\omega_{;p}\,\omega_{;a}\,\omega_{;qb}-34\,\omega_{;pa}\,\omega_{;qb}\\
&\quad+63\,\omega_{;p}\,\omega_{;q}\,\omega_{;ab}-36\,\omega_{;p}\,\omega_{;qab})\,R^{pq}+672\,\omega_{;p}\,\omega_{;q}\,R_{ra}\,R_b{}^{pqr}\\
&-8\,(179\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;r}-165\,\omega_{;r}\,\omega^{;r}\,\omega_{;pq}+378\,\omega_{;q}\,\omega_{;r}\,\omega_{;p}{}^r\\
&\quad-8\,\omega_{;p}\,\omega_{;r}\,\omega_{;q}{}^r+8\,\omega_{;pr}\,\omega_{;q}{}^r-175\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^r-7\,\omega_{;pq}\,\omega_{;r}{}^r\\
&\quad-33\,\omega_{;r}\,\omega_{;pq}{}^r-105\,\omega_{;p}\,\omega_{;r}{}^r{}_q+45\,\omega_{;r}\,\omega_{;s}\,R_p{}^r{}_q{}^s)\,R_a{}^p{}_b{}^q\\
&4\,(1124\,\omega_{;q}\,\omega_{;a}\,\omega_{;pr}+400\,\omega_{;p}\,\omega_{;q}\,\omega_{;ra}-104\,\omega_{;pq}\,\omega_{;ra}\\
&\quad+174\,\omega_{;q}\,\omega_{;pra}-87\,\omega_{;q}\,\omega_{;s}\,R_{par}{}^s-87\,\omega_{;q}\,\omega_{;s}\,R_p{}^s{}_{ra})\,R_b{}^{pqr}
\end{aligned}
\tag{A17c}
$$

$$
\begin{aligned}
362880\bar{\Delta}^{(6')} = {}& 4792\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,\omega^{;q}\,\omega^{;r}-3732\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}\,\omega_{;r}{}^r\\
&+8808\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,\omega^{;qr}-84\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^q\,\omega_{;r}{}^r\\
&-2568\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^r\,\omega^{;pq}+1440\,\omega_{;p}\,\omega^{;p}\,\omega_{;qr}\,\omega^{;qr}\\
&+1248\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^p\,\omega^{;qr}-828\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;r}{}^{rq}\\
&+360\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;pqr}+35\,\omega_{;p}{}^p\,\omega_{;q}{}^q\,\omega_{;r}{}^r+84\,\omega_{;pq}\,\omega_{;r}{}^r\,\omega^{;pq}\\
&+-64\,\omega_{;pq}\,\omega_{;r}{}^p\,\omega^{;qr}+252\,\omega_{;p}\,\omega_{;q}{}^q\,\omega_{;r}{}^{rp}+306\,\omega_{;p}\,\omega_{;q}{}^p\,\omega_{;r}{}^{rq}\\
&+792\,\omega_{;p}\,\omega_{;qr}\,\omega^{;qrp}+162\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{rpq}\\
&+18\,\omega_{;p}\,\left(56\,\omega_{;q}\,\omega_{;r}\,\omega^{;r}+18\,\omega_{;r}\,\omega_{;q}{}^r+7\,\omega_{;q}\,\omega_{;r}{}^r\right)\,R^{pq}\\
&+108\,\omega_{;p}\,\omega_{;q}\,\left(\omega_{;r}\,R^{pq;r}-10\,\omega_{;rs}\,R^{prqs}\right)
\end{aligned}
\tag{A17d}
$$

APPENDIX B: EXPANSION TENSORS FOR $G_{\rm div,ren}$

We present the explicit expressions for the expansion tensors for $G_{\rm div,ren}$. The expansion scalars in the body are related to these tensors via $G^{(n)}_{\rm div,ren}=2^{\frac{n}{2}}p^{a_1}\cdots p^{a_n}G^{(n)}_{{\rm div,ren},a_1\cdots a_n}$.

$$
G^{(0)}_{\rm div,ren} = \frac{1}{6}\left(\omega_{;p}{}^p-\omega_{;p}\,\omega^{;p}\right)
\tag{B1a}
$$

$$
G^{(1)}_{{\rm div,ren},a} = \frac{1}{12}\left(2\,\omega_{;p}\,\omega_{;a}{}^p-\omega_{;p}{}^p{}_a\right)
\tag{B1b}
$$

$$
\begin{aligned}
G^{(2)}_{{\rm div,ren},ab} = {}&(4\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}+15\,\omega_{;p}\,R_{ab}{}^{;p}-15\,\omega_{;p}\,R_a{}^p{}_{;b}-4\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^p\\
&+2\,\omega_{;p}{}^p\,\omega_{;ab}+8\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^p-20\,\omega_{;pa}\,\omega_{;b}{}^p+6\,\omega_{;a}\,\omega_{;pb}{}^p\\
&-12\,\omega_{;a}\,\omega_{;p}{}^p{}_b+36\,\omega_{;p}\,\omega_{;ab}{}^p-54\,\omega_{;p}\,\omega_{;a}{}^p{}_b-12\,\omega_{;pab}{}^p\\
&+3\,\omega_{;pa}{}^p{}_b+9\,\omega_{;p}{}^p{}_{ab}+9\,\omega_{;abp}{}^p-5\,\omega_{;p}\,\omega^{;p}\,R_{ab}\\
&+5\,\omega_{;p}{}^p\,R_{ab}-10\,\omega_{;p}\,\omega_{;a}\,R_b{}^p+5\,\omega_{;pa}\,R_b{}^p+34\,\omega_{;p}\,\omega_{;q}\,R_a{}^q{}_b{}^p\\
&+10\,\omega_{;pq}\,R_a{}^q{}_b{}^p)\,/360\\
&-g_{ab}\,(2\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}+4\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^q-5\,\omega_{;p}{}^p\,\omega_{;q}{}^q+16\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}\\
&-4\,\omega_{;pq}\,\omega^{;pq}-12\,\omega_{;p}\,\omega_{;q}{}^{qp}-6\,\omega_{;p}\,\omega_{;q}\,R^{pq})\,/720
\end{aligned}
\tag{B1c}
$$

$$
\begin{aligned}
G^{(3)}_{{\rm div,ren},abc} = {}&(25\,\omega_{;p}\,\omega^{;p}\,R_{ab;c}+50\,\omega_{;p}\,\omega_{;a}\,R_{bc}{}^{;p}-10\,\omega_{;p}\,\omega_{;a}\,R_b{}^p{}_{;c}\\
&-130\,\omega_{;p}\,\omega_{;q}\,R_a{}^q{}_b{}^p{}_{;c}-25\,R_{ab;c}\,\omega_{;p}{}^p-40\,\omega_{;p}\,\omega_{;a}\,\omega^{;p}\,\omega_{;bc}\\
&+40\,\omega_{;a}\,\omega_{;p}{}^p\,\omega_{;bc}-52\,\omega_{;p}\,\omega_{;a}{}^p\,\omega_{;bc}\\
&-40\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}{}^p+80\,R_{pa;b}\,\omega_{;c}{}^p-100\,R_{ab;p}\,\omega_{;c}{}^p-40\,\omega_{;a}\,\omega_{;pb}\,\omega_{;c}{}^p\\
&+12\,\omega_{;p}\,\omega_{;ab}\,\omega_{;c}{}^p-40\,R_{paqb;c}\,\omega^{;pq}-21\,\omega_{;p}\,R_{ab;c}{}^p-39\,\omega_{;p}\,R_{ab}{}^{;p}{}_c\\
&+60\,\omega_{;p}\,R_a{}^p{}_{;bc}+20\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc}{}^p-40\,\omega_{;ab}\,\omega_{;pc}{}^p+60\,\omega_{;ab}\,\omega_{;p}{}^p{}_c\\
&-10\,\omega_{;p}{}^p\,\omega_{;abc}+20\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}{}^p-140\,\omega_{;pa}\,\omega_{;bc}{}^p-60\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^p{}_c\\
&+280\,\omega_{;pa}\,\omega_{;b}{}^p{}_c-80\,\omega_{;a}\,\omega_{;pbc}{}^p+10\,\omega_{;a}\,\omega_{;pb}{}^p{}_c+50\,\omega_{;a}\,\omega_{;p}{}^p{}_{bc}\\
&-110\,\omega_{;p}\,\omega_{;abc}{}^p-30\,\omega_{;p}\,\omega_{;ab}{}^p{}_c+180\,\omega_{;p}\,\omega_{;a}{}^p{}_{bc}+50\,\omega_{;a}\,\omega_{;bcp}{}^p\\
&+20\,\omega_{;pabc}{}^p+20\,\omega_{;pab}{}^p{}_c-10\,\omega_{;pa}{}^p{}_{bc}-20\,\omega_{;p}{}^p{}_{abc}\\
&+10\,\omega_{;abpc}{}^p-20\,\omega_{;abp}{}^p{}_c-20\,\omega_{;abcp}{}^p+50\,\omega_{;p}\,\omega_{;a}{}^p\,R_{bc}\\
&-25\,\omega_{;p}{}^p{}_a\,R_{bc}+10\,\omega_{;p}\,R_a{}^p\,R_{bc}-20\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,R_c{}^p+70\,\omega_{;a}\,\omega_{;pb}\,R_c{}^p
\end{aligned}
$$

$$
\begin{aligned}
&+60\,\omega_{;p}\,\omega_{;ab}\,R_c{}^p - 50\,\omega_{;pab}\,R_c{}^p + 20\,\omega_{;abp}\,R_c{}^p - 10\,\omega_{;p}\,R_{ab}\,R_c{}^p\\
&+20\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,R_b{}^q{}_c{}^p + 20\,\omega_{;a}\,\omega_{;pq}\,R_b{}^q{}_c{}^p\\
&-480\,\omega_{;p}\,\omega_{;qa}\,R_b{}^q{}_c{}^p + 20\,\omega_{;pqa}\,R_b{}^q{}_c{}^p\\
&-80\,\omega_{;paq}\,R_b{}^q{}_c{}^p - 32\,\omega_{;p}\,R_{qa}\,R_b{}^q{}_c{}^p\\
&-32\,\omega_{;p}\,R_{qra}{}^p\,R_b{}^r{}_c{}^q + 24\,\omega_{;p}\,R_{qar}{}^p\,R_b{}^r{}_c{}^q\\
&+32\,\omega_{;p}\,R_q{}^p{}_{ra}\,R_b{}^r{}_c{}^q - 16\,\omega_{;p}\,R_{qarb}\,R_c{}^{qpr})\,/3600\\
&-g_{ab}\,(15\,\omega_{;p}\,\omega_{;q}\,R_c{}^{p;q} - 100\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;c}{}^p\\
&-20\,\omega_{;p}\,\omega_{;q}{}^q\,\omega_{;c}{}^p + 80\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;c}{}^q\\
&+132\,\omega_{;p}\,\omega_{;q}{}^p\,\omega_{;c}{}^q - 212\,\omega_{;p}\,\omega_{;qc}\,\omega^{;pq}\\
&-10\,\omega_{;p}\,\omega^{;p}\,\omega_{;qc}{}^q + 10\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{pq}\\
&+25\,\omega_{;p}{}^p\,\omega_{;q}{}^q{}_c - 10\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{qp} + 30\,\omega_{;pc}\,\omega_{;q}{}^{qp} - 70\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}{}^{pq}\\
&+10\,\omega_{;pq}\,\omega_{;c}{}^{pq} + 30\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}{}_c + 10\,\omega_{;pq}\,\omega^{;pq}{}_c - 20\,\omega_{;p}\,\omega_{;qc}{}^{pq}\\
&+5\,\omega_{;p}\,\omega_{;qc}{}^{qp} + 10\,\omega_{;p}\,\omega_{;q}{}^p{}_c{}^q - 5\,\omega_{;p}\,\omega_{;q}{}^{pq}{}_c + 25\,\omega_{;p}\,\omega_{;q}{}^q{}_c{}^p\\
&+5\,\omega_{;p}\,\omega_{;q}{}^{qp}{}_c + 10\,\omega_{;p}\,\omega_{;c}{}^p{}_q{}^q + 10\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,R_c{}^q - 5\,\omega_{;p}\,\omega_{;q}{}^p\,R_c{}^q\\
&-10\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,R^{pq} + 35\,\omega_{;p}\,\omega_{;qc}\,R^{pq})\,/3600
\end{aligned}
\tag{B1d}
$$

$$
\begin{aligned}
G^{(4)}_{\mathrm{div,ren},abcd} \;=\; &(-960\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega^{;p} + 2520\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,R_c{}^p{}_{;d}\\
&-4320\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,R_b{}^q{}_c{}^p{}_{;d} + 960\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}\,\omega_{;p}{}^p\\
&-1440\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}\,\omega_{;cd} - 6300\,\omega_{;p}\,R_{ab}{}^{;p}\,\omega_{;cd}\\
&+480\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^p\,\omega_{;cd} + 2760\,\omega_{;p}\,\omega^{;p}\,\omega_{;ab}\,\omega_{;cd}\\
&-2640\,\omega_{;p}{}^p\,\omega_{;ab}\,\omega_{;cd} - 48544\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^p\,\omega_{;cd}\\
&+7384\,\omega_{;pa}\,\omega_{;b}{}^p\,\omega_{;cd} - 960\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;d}{}^p\\
&+5040\,\omega_{;a}\,R_{pb;c}\,\omega_{;d}{}^p - 6300\,\omega_{;p}\,R_{ab;c}\,\omega_{;d}{}^p\\
&-12600\,\omega_{;a}\,R_{bc;p}\,\omega_{;d}{}^p + 3840\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc}\,\omega_{;d}{}^p\\
&+57664\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}\,\omega_{;d}{}^p - 2104\,\omega_{;pa}\,\omega_{;bc}\,\omega_{;d}{}^p\\
&-47520\,\omega_{;p}\,R_{qab}{}^p{}_{;c}\,\omega_{;d}{}^q - 2520\,\omega_{;a}\,R_{pbqc;d}\,\omega^{;pq}\\
&-1890\,\omega_{;p}\,\omega^{;p}\,R_{ab;cd} + 1890\,\omega_{;p}{}^p\,R_{ab;cd}\\
&+588\,\omega_{;p}\,\omega_{;a}\,R_{bc;d}{}^p + 2352\,\omega_{;pa}\,R_{bc;d}{}^p\\
&-5628\,\omega_{;p}\,\omega_{;a}\,R_{bc}{}^{;p}{}_d + 7728\,\omega_{;pa}\,R_{bc}{}^{;p}{}_d + 2520\,\omega_{;p}\,\omega_{;a}\,R_b{}^p{}_{;cd}\\
&-8820\,\omega_{;pa}\,R_b{}^p{}_{;cd} + 7920\,\omega_{;p}\,\omega_{;q}\,R_a{}^q{}_b{}^p{}_{;cd} + 2520\,\omega_{;pq}\,R_a{}^q{}_b{}^p{}_{;cd}\\
&-720\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;pd}{}^p - 6480\,\omega_{;a}\,\omega_{;bc}\,\omega_{;pd}{}^p + 2160\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,\omega_{;p}{}^p{}_d\\
&+3150\,R_{ab;c}\,\omega_{;p}{}^p{}_d + 1800\,\omega_{;a}\,\omega_{;bc}\,\omega_{;p}{}^p{}_d + 2880\,\omega_{;p}\,\omega_{;a}\,\omega^{;p}\,\omega_{;bcd}\\
&-3240\,\omega_{;a}\,\omega_{;p}{}^p\,\omega_{;bcd} + 2880\,\omega_{;p}\,\omega_{;a}{}^p\,\omega_{;bcd} + 3600\,\omega_{;pa}{}^p\,\omega_{;bcd}\\
&-4500\,\omega_{;p}{}^p{}_a\,\omega_{;bcd} - 6480\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;cd}{}^p - 5040\,R_{pa;b}\,\omega_{;cd}{}^p\\
&+3150\,R_{ab;p}\,\omega_{;cd}{}^p - 10080\,\omega_{;a}\,\omega_{;pb}\,\omega_{;cd}{}^p - 360\,\omega_{;p}\,\omega_{;ab}\,\omega_{;cd}{}^p\\
&-9000\,\omega_{;pab}\,\omega_{;cd}{}^p + 8730\,\omega_{;abp}\,\omega_{;cd}{}^p + 9360\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}{}^p{}_d\\
&-1260\,R_{pa;b}\,\omega_{;c}{}^p{}_d + 6300\,R_{ab;p}\,\omega_{;c}{}^p{}_d + 19440\,\omega_{;a}\,\omega_{;pb}\,\omega_{;c}{}^p{}_d\\
&+5760\,\omega_{;p}\,\omega_{;ab}\,\omega_{;c}{}^p{}_d - 6480\,\omega_{;pab}\,\omega_{;c}{}^p{}_d + 3780\,R_{paqb;c}\,\omega_{;d}{}^{pq}\\
&+3780\,R_{paqb;c}\,\omega_{;d}{}^{qp} - 1260\,R_{paqb;c}\,\omega^{;pq}{}_d + 1470\,\omega_{;p}\,R_{ab;cd}{}^p\\
&+462\,\omega_{;p}\,R_{ab;c}{}^p{}_d + 798\,\omega_{;p}\,R_{ab}{}^{;p}{}_{cd} - 2730\,\omega_{;p}\,R_a{}^p{}_{;bcd}\\
&-1050\,\omega_{;p}\,R_{qab}{}^p{}_{;cd}{}^q - 2880\,\omega_{;a}\,\omega_{;b}\,\omega_{;pcd}{}^p + 11520\,\omega_{;ab}\,\omega_{;pcd}{}^p\\
&-360\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc}{}^p{}_d - 1080\,\omega_{;ab}\,\omega_{;pc}{}^p{}_d + 1080\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^p{}_{cd}\\
&-6840\,\omega_{;ab}\,\omega_{;p}{}^p{}_{cd} + 720\,\omega_{;p}{}^p\,\omega_{;abcd} - 360\,\omega_{;p}\,\omega_{;a}\,\omega_{;bcd}{}^p\\
&+10800\,\omega_{;pa}\,\omega_{;bcd}{}^p - 3960\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}{}^p{}_d + 4320\,\omega_{;pa}\,\omega_{;bc}{}^p{}_d\\
&+7200\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^p{}_{cd} - 23760\,\omega_{;pa}\,\omega_{;b}{}^p{}_{cd} + 1080\,\omega_{;a}\,\omega_{;b}\,\omega_{;cdp}{}^p\\
&-6840\,\omega_{;ab}\,\omega_{;cdp}{}^p + 3600\,\omega_{;a}\,\omega_{;pbcd}{}^p + 3600\,\omega_{;a}\,\omega_{;pbc}{}^p{}_d\\
&-1440\,\omega_{;a}\,\omega_{;pb}{}^p{}_{cd} - 3240\,\omega_{;a}\,\omega_{;p}{}^p{}_{bcd} + 6120\,\omega_{;p}\,\omega_{;abcd}{}^p\\
&+2880\,\omega_{;p}\,\omega_{;abc}{}^p{}_d - 360\,\omega_{;p}\,\omega_{;ab}{}^p{}_{cd} - 10440\,\omega_{;p}\,\omega_{;a}{}^p{}_{bcd}\\
&+1800\,\omega_{;a}\,\omega_{;bcpd}{}^p - 3240\,\omega_{;a}\,\omega_{;bcp}{}^p{}_d - 3240\,\omega_{;a}\,\omega_{;bcdp}{}^p\\
&-720\,\omega_{;pabcd}{}^p - 720\,\omega_{;pabc}{}^p{}_d - 720\,\omega_{;pab}{}^p{}_{cd}\\
&+540\,\omega_{;pa}{}^p{}_{bcd} + 900\,\omega_{;p}{}^p{}_{abcd} - 360\,\omega_{;abpcd}{}^p\\
&-360\,\omega_{;abpc}{}^p{}_d + 900\,\omega_{;abp}{}^p{}_{cd} - 360\,\omega_{;abcpd}{}^p\\
&+900\,\omega_{;abcp}{}^p{}_d + 900\,\omega_{;abcdp}{}^p + 840\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega^{;p}\,R_{cd}\\
&+3150\,\omega_{;p}\,R_{ab}{}^{;p}\,R_{cd} - 3150\,\omega_{;p}\,R_a{}^p{}_{;b}\,R_{cd} - 840\,\omega_{;a}\,\omega_{;b}\,\omega_{;p}{}^p\,R_{cd}\\
&+420\,\omega_{;p}{}^p\,\omega_{;ab}\,R_{cd} + 1680\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}{}^p\,R_{cd} - 4200\,\omega_{;pa}\,\omega_{;b}{}^p\,R_{cd}\\
&+1260\,\omega_{;a}\,\omega_{;pb}{}^p\,R_{cd} - 2520\,\omega_{;a}\,\omega_{;p}{}^p{}_b\,R_{cd} + 7560\,\omega_{;p}\,\omega_{;ab}{}^p\,R_{cd}\\
&-11340\,\omega_{;p}\,\omega_{;a}{}^p{}_b\,R_{cd} - 2520\,\omega_{;pab}{}^p\,R_{cd} + 630\,\omega_{;pa}{}^p{}_b\,R_{cd}\\
&+1890\,\omega_{;p}{}^p{}_{ab}\,R_{cd} + 1890\,\omega_{;abp}{}^p\,R_{cd} - 525\,\omega_{;p}\,\omega^{;p}\,R_{ab}\,R_{cd}
\end{aligned}
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\begin{aligned}
&+525\,\omega_{;p}{}^{p}\,R_{ab}\,R_{cd}-980\,\omega_{;p}\,\omega_{;a}\,R_{b}{}^{p}\,R_{cd}-770\,\omega_{;pa}\,R_{b}{}^{p}\,R_{cd}\\
&+1680\,\omega_{;p}\,\omega_{;a}\,\omega_{;b}\,\omega_{;c}\,R_{d}{}^{p}+4200\,\omega_{;a}\,\omega_{;b}\,\omega_{;pc}\,R_{d}{}^{p}\\
&+6720\,\omega_{;p}\,\omega_{;a}\,\omega_{;bc}\,R_{d}{}^{p}\\
&-10920\,\omega_{;pa}\,\omega_{;bc}\,R_{d}{}^{p}-10080\,\omega_{;a}\,\omega_{;pbc}\,R_{d}{}^{p}-5040\,\omega_{;p}\,\omega_{;abc}\,R_{d}{}^{p}\\
&+2520\,\omega_{;a}\,\omega_{;bcp}\,R_{d}{}^{p}+3780\,\omega_{;pabc}\,R_{d}{}^{p}-1260\,\omega_{;abcp}\,R_{d}{}^{p}\\
&-1120\,\omega_{;p}\,\omega_{;a}\,R_{bc}\,R_{d}{}^{p}+1820\,\omega_{;pa}\,R_{bc}\,R_{d}{}^{p}-1764\,\omega_{;p}\,R_{qab}{}^{p}{}_{;c}\,R_{d}{}^{q}\\
&-480\,\omega_{;p}\,\omega_{;q}\,R_{rba}{}^{q}\,R_{c}{}^{p}{}_{d}{}^{r}-6480\,\omega_{;p}\,\omega_{;q}\,\omega_{;a}\,\omega_{;b}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+714\,\omega_{;p}\,R_{qa;b}\,R_{c}{}^{q}{}_{d}{}^{p}+5250\,\omega_{;p}\,R_{ab;q}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&-1680\,\omega_{;a}\,\omega_{;b}\,\omega_{;pq}\,R_{c}{}^{q}{}_{d}{}^{p}-15360\,\omega_{;p}\,\omega_{;a}\,\omega_{;qb}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+34800\,\omega_{;pa}\,\omega_{;qb}\,R_{c}{}^{q}{}_{d}{}^{p}-240\,\omega_{;p}\,\omega_{;q}\,\omega_{;ab}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&-1680\,\omega_{;pq}\,\omega_{;ab}\,R_{c}{}^{q}{}_{d}{}^{p}+5040\,\omega_{;a}\,\omega_{;pqb}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&-10080\,\omega_{;a}\,\omega_{;pbq}\,R_{c}{}^{q}{}_{d}{}^{p}+20160\,\omega_{;p}\,\omega_{;qab}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+32040\,\omega_{;p}\,\omega_{;abq}\,R_{c}{}^{q}{}_{d}{}^{p}-2520\,\omega_{;pqab}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+5040\,\omega_{;pabq}\,R_{c}{}^{q}{}_{d}{}^{p}+2520\,\omega_{;abpq}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+2016\,\omega_{;p}\,\omega_{;a}\,R_{qb}\,R_{c}{}^{q}{}_{d}{}^{p}+3024\,\omega_{;pa}\,R_{qb}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+7140\,\omega_{;p}\,\omega_{;q}\,R_{ab}\,R_{c}{}^{q}{}_{d}{}^{p}+2100\,\omega_{;pq}\,R_{ab}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&-3150\,\omega_{;p}\,R_{qab}{}^{p}{}_{;r}\,R_{c}{}^{q}{}_{d}{}^{r}-16320\,\omega_{;p}\,\omega_{;q}\,R_{rba}{}^{q}\,R_{c}{}^{r}{}_{d}{}^{p}\\
&-6720\,\omega_{;pq}\,R_{rba}{}^{q}\,R_{c}{}^{r}{}_{d}{}^{p}-2016\,\omega_{;p}\,R_{qra}{}^{p}{}_{;b}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&+420\,\omega_{;p}\,R_{qarb}{}^{;p}\,R_{c}{}^{r}{}_{d}{}^{q}-3990\,\omega_{;p}\,R_{q}{}^{p}{}_{ra;b}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&-3584\,\omega_{;p}\,\omega_{;a}\,R_{qrb}{}^{p}\,R_{c}{}^{r}{}_{d}{}^{q}+5824\,\omega_{;pa}\,R_{qrb}{}^{p}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&-420\,\omega_{;p}\,\omega^{;p}\,R_{qarb}\,R_{c}{}^{r}{}_{d}{}^{q}+420\,\omega_{;p}{}^{p}\,R_{qarb}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&+2688\,\omega_{;p}\,\omega_{;a}\,R_{qbr}{}^{p}\,R_{c}{}^{r}{}_{d}{}^{q}-4368\,\omega_{;pa}\,R_{qbr}{}^{p}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&+3584\,\omega_{;p}\,\omega_{;a}\,R_{q}{}^{p}{}_{rb}\,R_{c}{}^{r}{}_{d}{}^{q}-5824\,\omega_{;pa}\,R_{q}{}^{p}{}_{rb}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&-3276\,\omega_{;p}\,R_{qarb;c}\,R_{d}{}^{pqr}-3318\,\omega_{;p}\,R_{qarb;c}\,R_{d}{}^{qpr}\\
&-2912\,\omega_{;p}\,\omega_{;a}\,R_{qbrc}\,R_{d}{}^{qpr}+3472\,\omega_{;pa}\,R_{qbrc}\,R_{d}{}^{qpr}\\
&-672\,\omega_{;p}\,R_{qarb;c}\,R_{d}{}^{rpq}\big)/907200
\end{aligned}
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\begin{aligned}
&+g_{ab}\big(240\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega_{;d}\,\omega^{;p}\,\omega^{;q}+1050\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,R_{cd}{}^{;p}\\
&-1050\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,R_{cd}{}^{;q}-1050\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,R_{c}{}^{p}{}_{;d}\\
&+1470\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,R_{c}{}^{q}{}_{;d}-1440\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,R_{d}{}^{p;q}\\
&+480\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,R^{pq}{}_{;d}+225\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,R_{c}{}^{p}{}_{d}{}^{q;r}\\
&+1995\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,R_{c}{}^{r}{}_{d}{}^{p;q}+80\,\omega_{;p}\,\omega_{;c}\,\omega_{;d}\,\omega^{;p}\,\omega_{;q}{}^{q}\\
&+1050\,\omega_{;p}\,R_{cd}{}^{;p}\,\omega_{;q}{}^{q}-1050\,\omega_{;p}\,R_{c}{}^{p}{}_{;d}\,\omega_{;q}{}^{q}\\
&-280\,\omega_{;c}\,\omega_{;d}\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}-240\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}\,\omega_{;cd}\\
&+280\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}\,\omega_{;cd}+140\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}\,\omega_{;cd}\\
&+1680\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega^{;q}\,\omega_{;d}{}^{p}-640\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{q}\,\omega_{;d}{}^{p}\\
&-1800\,\omega_{;pc}\,\omega_{;q}{}^{q}\,\omega_{;d}{}^{p}-760\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}{}^{q}\,\omega_{;d}{}^{p}-3360\,\omega_{;pq}\,\omega_{;c}{}^{q}\,\omega_{;d}{}^{p}\\
&-720\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega^{;p}\,\omega_{;d}{}^{q}+1620\,\omega_{;p}\,R_{qc}{}^{;p}\,\omega_{;d}{}^{q}-240\,\omega_{;p}\,R_{q}{}^{p}{}_{;c}\,\omega_{;d}{}^{q}\\
&+360\,\omega_{;p}\,R_{c}{}^{p}{}_{;q}\,\omega_{;d}{}^{q}-320\,\omega_{;p}\,\omega^{;p}\,\omega_{;qc}\,\omega_{;d}{}^{q}+1120\,\omega_{;p}{}^{p}\,\omega_{;qc}\,\omega_{;d}{}^{q}\\
&-2936\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{p}\,\omega_{;d}{}^{q}+1280\,\omega_{;pc}\,\omega_{;q}{}^{p}\,\omega_{;d}{}^{q}\\
&+640\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega_{;d}\,\omega^{;pq}\\
&-1260\,\omega_{;p}\,R_{qc;d}\,\omega^{;pq}+1050\,\omega_{;p}\,R_{cd;q}\,\omega^{;pq}-200\,\omega_{;c}\,\omega_{;d}\,\omega_{;pq}\,\omega^{;pq}\\
&+3576\,\omega_{;p}\,\omega_{;c}\,\omega_{;qd}\,\omega^{;pq}+80\,\omega_{;p}\,\omega_{;q}\,\omega_{;cd}\,\omega^{;pq}+160\,\omega_{;pq}\,\omega_{;cd}\,\omega^{;pq}\\
&+420\,\omega_{;p}\,R_{qcrd}{}^{;p}\,\omega^{;qr}-900\,\omega_{;p}\,R_{qcd}{}^{p}{}_{;r}\,\omega^{;qr}-60\,\omega_{;p}\,R_{q}{}^{p}{}_{rc;d}\,\omega^{;qr}\\
&+630\,\omega_{;p}\,\omega_{;q}\,R_{cd}{}^{;pq}+360\,\omega_{;p}\,\omega_{;q}\,R_{cd}{}^{;qp}-924\,\omega_{;p}\,\omega_{;q}\,R_{c}{}^{p}{}_{;d}{}^{q}\\
&+384\,\omega_{;p}\,\omega_{;q}\,R_{c}{}^{p;q}{}_{d}-360\,\omega_{;p}\,\omega_{;q}\,R_{c}{}^{q}{}_{;d}{}^{p}+270\,\omega_{;p}\,\omega_{;q}\,R^{pq}{}_{;cd}\\
&+360\,\omega_{;p}\,\omega_{;c}\,\omega^{;p}\,\omega_{;qd}{}^{q}+420\,\omega_{;c}\,\omega_{;p}{}^{p}\,\omega_{;qd}{}^{q}-600\,\omega_{;p}\,\omega_{;c}{}^{p}\,\omega_{;qd}{}^{q}\\
&-240\,\omega_{;p}\,\omega_{;c}\,\omega_{;d}\,\omega_{;q}{}^{pq}+480\,\omega_{;c}\,\omega_{;pd}\,\omega_{;q}{}^{pq}+480\,\omega_{;p}\,\omega_{;cd}\,\omega_{;q}{}^{pq}\\
&-240\,\omega_{;p}\,\omega_{;c}\,\omega^{;p}\,\omega_{;q}{}^{q}{}_{d}-840\,\omega_{;c}\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}{}_{d}-720\,\omega_{;p}\,\omega_{;c}{}^{p}\,\omega_{;q}{}^{q}{}_{d}\\
&+525\,\omega_{;p}{}^{p}{}_{c}\,\omega_{;q}{}^{q}{}_{d}-360\,\omega_{;p}\,\omega_{;c}\,\omega_{;d}\,\omega_{;q}{}^{qp}-1080\,\omega_{;c}\,\omega_{;pd}\,\omega_{;q}{}^{qp}\\
&-60\,\omega_{;p}\,\omega_{;cd}\,\omega_{;q}{}^{qp}+420\,\omega_{;pcd}\,\omega_{;q}{}^{qp}+1260\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;cd}{}^{p}\\
&+3720\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;cd}{}^{p}-2340\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;cd}{}^{q}+3360\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;cd}{}^{q}\\
&+750\,\omega_{;p}{}^{p}{}_{q}\,\omega_{;cd}{}^{q}-3780\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;c}{}^{p}{}_{d}-4140\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;c}{}^{p}{}_{d}\\
&+4380\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;c}{}^{q}{}_{d}-5160\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;c}{}^{q}{}_{d}-420\,\omega_{;p}{}^{p}{}_{q}\,\omega_{;c}{}^{q}{}_{d}\\
&+480\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega_{;d}{}^{pq}-4320\,\omega_{;p}\,\omega_{;qc}\,\omega_{;d}{}^{pq}+1500\,\omega_{;pcq}\,\omega_{;d}{}^{pq}\\
&-2040\,\omega_{;p}\,\omega_{;qc}\,\omega_{;d}{}^{qp}-1260\,\omega_{;pcq}\,\omega_{;d}{}^{qp}-240\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega^{;pq}{}_{d}\\
&-360\,\omega_{;c}\,\omega_{;pq}\,\omega^{;pq}{}_{d}+2280\,\omega_{;p}\,\omega_{;qc}\,\omega^{;pq}{}_{d}+210\,\omega_{;pqc}\,\omega^{;pq}{}_{d}\\
&-480\,\omega_{;p}\,\omega^{;p}\,\omega_{;qcd}{}^{q}-840\,\omega_{;p}{}^{p}\,\omega_{;qcd}{}^{q}-60\,\omega_{;p}\,\omega^{;p}\,\omega_{;qc}{}^{q}{}_{d}
\end{aligned}
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\begin{aligned}
&+210\,\omega_{;p}{}^{p}\,\omega_{;qc}{}^{q}{}_{d}+1200\,\omega_{;p}\,\omega_{;c}\,\omega_{;qd}{}^{pq}-960\,\omega_{;pc}\,\omega_{;qd}{}^{pq}\\
&+240\,\omega_{;pc}\,\omega_{;qd}{}^{qp}-480\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{p}{}_{d}{}^{q}+240\,\omega_{;pc}\,\omega_{;q}{}^{p}{}_{d}{}^{q}\\
&+360\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{pq}{}_{d}-180\,\omega_{;pc}\,\omega_{;q}{}^{pq}{}_{d}+180\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}{}_{cd}\\
&+630\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}{}_{cd}-840\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{q}{}_{d}{}^{p}+960\,\omega_{;pc}\,\omega_{;q}{}^{q}{}_{d}{}^{p}\\
&-240\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{qp}{}_{d}+360\,\omega_{;pc}\,\omega_{;q}{}^{qp}{}_{d}+180\,\omega_{;p}\,\omega^{;p}\,\omega_{;cdq}{}^{q}\\
&+630\,\omega_{;p}{}^{p}\,\omega_{;cdq}{}^{q}+2220\,\omega_{;p}\,\omega_{;q}\,\omega_{;cd}{}^{pq}+480\,\omega_{;pq}\,\omega_{;cd}{}^{pq}\\
&-4440\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}{}^{p}{}_{d}{}^{q}-360\,\omega_{;pq}\,\omega_{;c}{}^{p}{}_{d}{}^{q}-480\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}{}^{pq}{}_{d}\\
&+120\,\omega_{;pq}\,\omega_{;c}{}^{pq}{}_{d}-480\,\omega_{;p}\,\omega_{;c}\,\omega_{;d}{}^{p}{}_{q}{}^{q}+660\,\omega_{;pc}\,\omega_{;d}{}^{p}{}_{q}{}^{q}\\
&+1740\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}{}_{cd}+240\,\omega_{;pq}\,\omega^{;pq}{}_{cd}-360\,\omega_{;p}\,\omega_{;qcd}{}^{pq}\\
&-600\,\omega_{;p}\,\omega_{;qcd}{}^{qp}-120\,\omega_{;p}\,\omega_{;qc}{}^{p}{}_{d}{}^{q}-120\,\omega_{;p}\,\omega_{;qc}{}^{pq}{}_{d}\\
&+240\,\omega_{;p}\,\omega_{;qc}{}^{q}{}_{d}{}^{p}+60\,\omega_{;p}\,\omega_{;qc}{}^{qp}{}_{d}+240\,\omega_{;p}\,\omega_{;q}{}^{p}{}_{cd}{}^{q}\\
&+240\,\omega_{;p}\,\omega_{;q}{}^{p}{}_{c}{}^{q}{}_{d}-180\,\omega_{;p}\,\omega_{;q}{}^{pq}{}_{cd}+540\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{cd}{}^{p}\\
&+240\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{c}{}^{p}{}_{d}-60\,\omega_{;p}\,\omega_{;q}{}^{qp}{}_{cd}-180\,\omega_{;p}\,\omega_{;cdq}{}^{pq}\\
&+540\,\omega_{;p}\,\omega_{;cdq}{}^{qp}+360\,\omega_{;p}\,\omega_{;cd}{}^{p}{}_{q}{}^{q}+240\,\omega_{;p}\,\omega_{;c}{}^{p}{}_{qd}{}^{q}\\
&-180\,\omega_{;p}\,\omega_{;c}{}^{p}{}_{q}{}^{q}{}_{d}-180\,\omega_{;p}\,\omega_{;c}{}^{p}{}_{dq}{}^{q}-70\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}\,R_{cd}\\
&-140\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}\,R_{cd}+175\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}\,R_{cd}-560\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}\,R_{cd}\\
&+140\,\omega_{;pq}\,\omega^{;pq}\,R_{cd}+420\,\omega_{;p}\,\omega_{;q}{}^{qp}\,R_{cd}-700\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega^{;q}\,R_{d}{}^{p}\\
&-700\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{q}\,R_{d}{}^{p}+350\,\omega_{;pc}\,\omega_{;q}{}^{q}\,R_{d}{}^{p}+420\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega^{;p}\,R_{d}{}^{q}\\
&+700\,\omega_{;p}\,\omega^{;p}\,\omega_{;qc}\,R_{d}{}^{q}-560\,\omega_{;p}\,\omega_{;c}\,\omega_{;q}{}^{p}\,R_{d}{}^{q}+280\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}{}^{p}\,R_{d}{}^{q}\\
&-140\,\omega_{;pq}\,\omega_{;c}{}^{p}\,R_{d}{}^{q}+840\,\omega_{;p}\,\omega_{;qc}{}^{p}\,R_{d}{}^{q}-420\,\omega_{;p}\,\omega_{;q}{}^{p}{}_{c}\,R_{d}{}^{q}\\
&-420\,\omega_{;p}\,\omega_{;c}{}^{p}{}_{q}\,R_{d}{}^{q}-40\,\omega_{;p}\,\omega_{;q}\,\omega_{;c}\,\omega_{;d}\,R^{pq}-1640\,\omega_{;p}\,\omega_{;c}\,\omega_{;qd}\,R^{pq}\\
&+920\,\omega_{;pc}\,\omega_{;qd}\,R^{pq}-240\,\omega_{;p}\,\omega_{;q}\,\omega_{;cd}\,R^{pq}+420\,\omega_{;p}\,\omega_{;qcd}\,R^{pq}\\
&+540\,\omega_{;p}\,\omega_{;cdq}\,R^{pq}+210\,\omega_{;p}\,\omega_{;q}\,R_{cd}\,R^{pq}-840\,\omega_{;p}\,\omega_{;qr}{}^{r}\,R_{c}{}^{p}{}_{d}{}^{q}\\
&+1800\,\omega_{;p}\,\omega_{;q}{}^{q}{}_{r}\,R_{c}{}^{p}{}_{d}{}^{r}+3820\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{r}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+700\,\omega_{;pq}\,\omega_{;r}{}^{r}\,R_{c}{}^{q}{}_{d}{}^{p}+840\,\omega_{;p}\,\omega_{;qr}{}^{r}\,R_{c}{}^{q}{}_{d}{}^{p}\\
&+2680\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{p}\,R_{c}{}^{q}{}_{d}{}^{r}+920\,\omega_{;pq}\,\omega_{;r}{}^{p}\,R_{c}{}^{q}{}_{d}{}^{r}\\
&+840\,\omega_{;p}\,\omega_{;q}{}^{p}{}_{r}\,R_{c}{}^{q}{}_{d}{}^{r}+864\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{p}\,R_{c}{}^{q}{}_{d}{}^{r}\\
&+700\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;q}\,R_{c}{}^{r}{}_{d}{}^{p}-168\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{q}\,R_{c}{}^{r}{}_{d}{}^{p}\\
&-2100\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,R_{c}{}^{r}{}_{d}{}^{q}-280\,\omega_{;p}\,\omega^{;p}\,\omega_{;qr}\,R_{c}{}^{r}{}_{d}{}^{q}\\
&+960\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{q}{}_{s}{}^{p}\,R_{c}{}^{r}{}_{d}{}^{s}-1560\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{q}{}_{s}{}^{p}\,R_{c}{}^{s}{}_{d}{}^{r}\\
&-3280\,\omega_{;p}\,\omega_{;q}\,\omega_{;rc}\,R_{d}{}^{pqr}-280\,\omega_{;p}\,\omega_{;qrc}\,R_{d}{}^{pqr}\\
&+2480\,\omega_{;p}\,\omega_{;qcr}\,R_{d}{}^{pqr}-1176\,\omega_{;p}\,\omega_{;q}\,R_{rc}\,R_{d}{}^{pqr}\\
&+324\,\omega_{;p}\,\omega_{;q}\,R_{rsc}{}^{q}\,R_{d}{}^{prs}+1120\,\omega_{;p}\,\omega_{;qcr}\,R_{d}{}^{qpr}\\
&+432\,\omega_{;p}\,\omega_{;q}\,R_{rc}\,R_{d}{}^{qpr}-320\,\omega_{;p}\,\omega_{;c}\,\omega_{;qr}\,R_{d}{}^{rpq}\\
&+400\,\omega_{;pc}\,\omega_{;qr}\,R_{d}{}^{rpq}-1360\,\omega_{;p}\,\omega_{;qcr}\,R_{d}{}^{rpq}\\
&-80\,\omega_{;p}\,\omega_{;q}\,R_{rsc}{}^{q}\,R_{d}{}^{rps}+40\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{q}{}_{sc}\,R_{d}{}^{rps}\\
&-792\,\omega_{;p}\,\omega_{;q}\,R_{rsc}{}^{q}\,R_{d}{}^{spr}\\
&+560\,\omega_{;p}\,\omega_{;q}\,R_{r}{}^{q}{}_{sc}\,R_{d}{}^{spr})\,/302400
\end{aligned}
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\begin{aligned}
&+g_{ab}g_{cd}\,(-12\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,\omega^{;q}\,\omega^{;r}\\
&+108\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,R^{pr;q}-60\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega^{;q}\,\omega_{;r}{}^{r}\\
&+21\,\omega_{;p}\,\omega^{;p}\,\omega_{;q}{}^{q}\,\omega_{;r}{}^{r}+35\,\omega_{;p}{}^{p}\,\omega_{;q}{}^{q}\,\omega_{;r}{}^{r}-12\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{r}\,\omega^{;pq}\\
&+84\,\omega_{;pq}\,\omega_{;r}{}^{r}\,\omega^{;pq}-588\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;q}\,\omega^{;pr}-366\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{q}\,\omega^{;pr}\\
&+64\,\omega_{;pq}\,\omega_{;r}{}^{q}\,\omega^{;pr}+252\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,\omega^{;qr}-12\,\omega_{;p}\,\omega^{;p}\,\omega_{;qr}\,\omega^{;qr}\\
&-36\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;r}{}^{qr}+252\,\omega_{;p}\,\omega_{;q}\,\omega^{;q}\,\omega_{;r}{}^{rp}+252\,\omega_{;p}\,\omega_{;q}{}^{q}\,\omega_{;r}{}^{rp}\\
&-216\,\omega_{;p}\,\omega_{;q}\,\omega^{;p}\,\omega_{;r}{}^{rq}+306\,\omega_{;p}\,\omega_{;q}{}^{p}\,\omega_{;r}{}^{rq}-108\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;prq}\\
&+108\,\omega_{;p}\,\omega_{;qr}\,\omega^{;prq}+108\,\omega_{;p}\,\omega_{;qr}\,\omega^{;qrp}-36\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{pqr}\\
&-18\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{prq}+162\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{rpq}+54\,\omega_{;p}\,\omega_{;q}\,\omega^{;pq}{}_{r}{}^{r}\\
&+126\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{r}\,R^{pq}+126\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;q}\,R^{pr}\\
&-108\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}\,\omega^{;p}\,R^{qr}\\
&+306\,\omega_{;p}\,\omega_{;q}\,\omega_{;r}{}^{p}\,R^{qr}+36\,\omega_{;p}\,\omega_{;q}\,\omega_{;rs}\,R^{prqs})\,/181440
\end{aligned}
\tag{B1e}
$$

APPENDIX C: EXPANSION TENSORS FOR $G_{\rm fin}$

We present the explicit expressions for the expansion tensors for $G_{\rm fin}$.

$$
\begin{aligned}
G^{(0)}_{\rm fin} &= \frac{\kappa^2}{6\,e^{2\,w}} && \text{(C1a)}\\
G^{(1)}_{{\rm fin},a} &= \frac{\kappa^2\,w_{;a}}{6\,e^{2\,w}} && \text{(C1b)}\\
G^{(2)}_{{\rm fin},ab} &= \frac{\kappa^2}{72\,e^{2\,w}}\left(8\,w_{;a}\,w_{;b} - 4\,w_{;ab} - 2\,w_{;p}\,w^{;p}\,g_{ab} + w_{;p}{}^{p}\,g_{ab} + R_{ab}\right)\\
&\quad + \frac{\kappa^4}{360\,e^{4\,w}}\left(4\,e^{2\,w}\,\delta^\tau_a\,\delta^\tau_b - g_{ab}\right) && \text{(C1c)}\\
G^{(3)}_{{\rm fin},abc} &= \frac{\kappa^2}{144\,e^{2\,w}}\left(8\,w_{;a}\,w_{;b}\,w_{;c} - R_{ab;c} - 12\,w_{;a}\,w_{;bc} + 2\,w_{;abc} - 4\,w_{;p}\,w_{;a}\,w^{;p}\,g_{bc}\right.\\
&\qquad \left. + \,2\,w_{;a}\,w_{;p}{}^{p}\,g_{bc} + 4\,w_{;p}\,w_{;a}{}^{p}\,g_{bc} - w_{;p}{}^{p}{}_{a}\,g_{bc} + 2\,w_{;a}\,R_{bc}\right)\\
&\quad + \frac{\kappa^4}{180\,e^{4\,w}}\left(2\,e^{2\,w}\,\Gamma^\tau{}_{ab}\,\delta^\tau_c + 2\,e^{2\,w}\,w_{;a}\,\delta^\tau_b\,\delta^\tau_c - w_{;a}\,g_{bc}\right) && \text{(C1d)}\\
G^{(3)}_{{\rm fin},abc} &= \frac{\kappa^2}{8640\,e^{2\,w}}\left(192\,w_{;a}\,w_{;b}\,w_{;c}\,w_{;d} - 60\,w_{;a}\,R_{bc;d} - 576\,w_{;a}\,w_{;b}\,w_{;cd}\right.\\
&\qquad +144\,w_{;ab}\,w_{;cd} + 18\,R_{ab;cd} + 192\,w_{;a}\,w_{;bcd} - 24\,w_{;abcd}\\
&\qquad +80\,w_{;a}\,w_{;b}\,R_{cd} - 40\,w_{;ab}\,R_{cd} + 5\,R_{ab}\,R_{cd} + 4\,R_{paqb}\,R_c{}^p{}_d{}^q\\
&\qquad -2g_{ab}\left\{72\,w_{;p}\,w_{;c}\,w_{;d}\,w^{;p} - 15\,w_{;p}\,R_{cd}{}^{;p} + 15\,w_{;p}\,R_c{}^p{}_{;d}\right.\\
&\qquad\qquad -36\,w_{;c}\,w_{;d}\,w_{;p}{}^{p} - 36\,w_{;p}\,w^{;p}\,w_{;cd} + 18\,w_{;p}{}^{p}\,w_{;cd}\\
&\qquad\qquad -144\,w_{;p}\,w_{;c}\,w_{;d}{}^{p} + 36\,w_{;pc}\,w_{;d}{}^{p} - 6\,w_{;c}\,w_{;pd}{}^{p}\\
&\qquad\qquad +42\,w_{;c}\,w_{;p}{}^{p}{}_{d} - 18\,w_{;p}\,w_{;cd}{}^{p} + 54\,w_{;p}\,w_{;c}{}^{p}{}_{d}\\
&\qquad\qquad +12\,w_{;pcd}{}^{p} - 3\,w_{;pc}{}^{p}{}_{d} - 9\,w_{;p}{}^{p}{}_{cd}\\
&\qquad\qquad -9\,w_{;cdp}{}^{p} + 10\,w_{;p}\,w^{;p}\,R_{cd} - 5\,w_{;p}{}^{p}\,R_{cd}\\
&\qquad\qquad +10\,w_{;p}\,w_{;c}\,R_d{}^p - 5\,w_{;pc}\,R_d{}^p - 10\,w_{;p}\,w_{;q}\,R_c{}^p{}_d{}^q\\
&\qquad\qquad \left. -10\,w_{;pq}\,R_c{}^q{}_d{}^p\right\}\\
&\qquad +g_{ab}g_{cd}\left\{12\,w_{;p}\,w_{;q}\,w^{;p}\,w^{;q} - 14\,w_{;p}\,w^{;p}\,w_{;q}{}^{q} + 5\,w_{;p}{}^{p}\,w_{;q}{}^{q}\right.\\
&\qquad\qquad -28\,w_{;p}\,w_{;q}\,w^{;pq} + 4\,w_{;pq}\,w^{;pq}\\
&\qquad\qquad \left.\left. +12\,w_{;p}\,w_{;q}{}^{qp} + 6\,w_{;p}\,w_{;q}\,R^{pq}\right\}\right)\\
&\quad + \frac{\kappa^4}{4320\,e^{4\,w}}\left(4\,e^{2\,w}\left\{3\,\Gamma^\tau{}_{ab}\,\Gamma^\tau{}_{cd} + 12\,\Gamma^\tau{}_{ab}\,w_{;c}\,\delta^\tau_d - 4\,\Gamma^\tau{}_{ab;c}\,\delta^\tau_d\right.\right.\\
&\qquad\qquad \left. +8\,w_{;a}\,w_{;b}\,\delta^\tau_c\,\delta^\tau_d - 4\,w_{;ab}\,\delta^\tau_c\,\delta^\tau_d + \delta^\tau_c\,\delta^\tau_d\,R_{ab}\right\}\\
&\qquad -g_{ab}\left\{28\,w_{;c}\,w_{;d} - 8\,w_{;cd} + R_{cd}\right.\\
&\qquad\qquad \left. 4\,e^{2\,w}\,\left(2\,w_{;p}\,w^{;p} - w_{;p}{}^{p}\right)\,\delta^\tau_c\,\delta^\tau_d R_{cd}\right\}\\
&\qquad \left. +g_{ab}g_{cd}\left\{3\,w_{;p}\,w^{;p} - w_{;p}{}^{p}\right\}\right)\\
&\quad + \frac{\kappa^6}{15120\,e^{6\,w}}\left(16\,e^{4\,w}\,\delta^\tau_a\,\delta^\tau_b\,\delta^\tau_c\,\delta^\tau_d - 12\,e^{2\,w}\,\delta^\tau_c\,\delta^\tau_d\,g_{ab} + g_{ab}\,g_{cd}\right) && \text{(C1e)}
\end{aligned}
$$

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